On Horrocks Theorem

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All rings are commutative Noetherian and modules are finitely generated.

Serre's Problem

Jean-Pierre Serre in his 1955-paper Faisceaux algébriques cohérents raised the following problem.
Serre's Problem: Let P be the projective module over $R = k[x_1, \ldots, x_n]$, polynomial ring in n variables over a field k. Then is P free?
This is indeed true for $R = k[x]$, since then R is a principal ideal domain. When Serre raised this question, this was the only case in which the answer to the above problem was known. The locally free nature of projective modules implies that they can be identified with vector bundles on certain affine varieties.

Seshadri independently proved this for the two-variable case.

Theorem (Seshadri)

Projective modules over $k[x, y]$, a polynomial ring in two variables over a field k, are free.

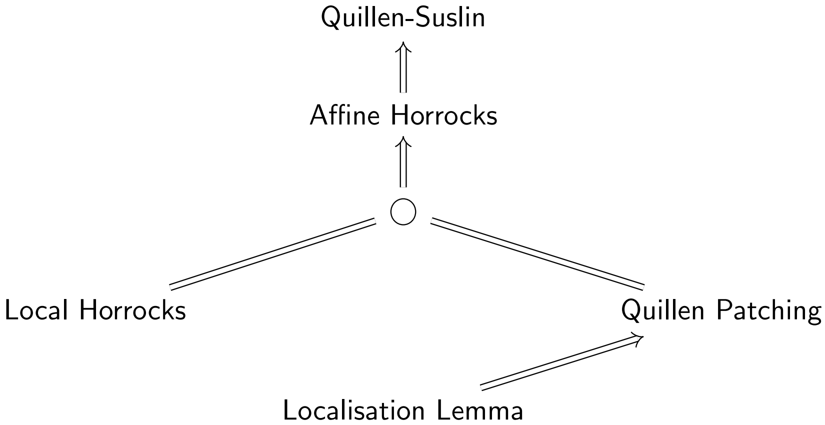
There were further developments to this problem until Quillen Suslin proved the following theorem

Theorem (Quillen-Suslin)

Let R be a principal ideal domain, $n \in \mathbb{N}$. Then any finitely generated projective module over $R[x_1, \ldots, x_n]$ is free

In other words: all vector bundles over $\mathbb{A}^n_R$ are trivial.

Overview of Quillen's Proof



Local Horrock's Theorem

We will discuss a geometric proof of the Horrocks theorem following Mohan Kumar's article: On a theorem of Seshadri.

Theorem (Geometric form)

Let R be a commutative Noetherian local ring. If a vector bundle $\mathcal{E}$ on $\mathbb{A}^1_R$ extends to a vector bundle on $\mathbb{P}^1_R$, then $\mathcal{E}$ is trivial.

Denote $R\langle x\rangle := S^{-1}R[x]$ the localisation of $R[x]$ at the multiplicative set S of all monic polynomials in x. Monic means leading coefficient 1. We write $M\langle x\rangle := S^{-1}M$ for an $R[x]$-module M.

Theorem (Algebraic Form)

Let R be a commutative Noetherian local ring, and P be a finitely generated projective module over $R[x]$. If $P\langle x\rangle = R\langle x\rangle \otimes_{R[x]} P$ is $R\langle x\rangle$ free, then P is $R[x]$-free.

Equivalence of the two forms

Lemma

For any commutative Noetherian local ring R, and P a finitely generated projective $R[x]$ module, then P extends to a vector bundle on $\mathbb{P}^1_R$ if and only if $P\langle x\rangle = R\langle x\rangle \otimes_{R[x]} P$ is $R\langle x\rangle$-free.

Proof: We will give the proof in one direction. Assume $P\langle x\rangle = R\langle x\rangle \otimes_{R[x]} P$ is $R\langle x\rangle$ free, then there exists a monic polynomial $f(x) \in R[x]$ such that $P_f = R[x]_f \otimes_{R[x]} P$ is $R[x]_f$-free. This means that the vector bundle $\widetilde{P}$, defined by P on Spec$R[x]$, has trivial restriction to Spec$R[x] \backslash V(f)$, where $V(f)$ is the closed set in Spec$R[x]$. We claim that $V(f)$ is actually closed in $\mathbb{P}^1_R$. If we can show this, then we can glue the bundle $\widetilde{P}$ on Spec$R[x]$ with a trivial bundle on the open set Spec$R[x^{-1}] \backslash V(f)$, to obtain a bundle on $\mathbb{P}^1_R$, as desired.

To show $V(f)$ is closed in $\mathbb{P}^1_R$, it is enough to show that $V(f) \cap \operatorname{Spec} R[x^{-1}]$ is closed in $\operatorname{Spec} R[x^{-1}]$. Define $g(x^{-1}) = x^{-n} f(x) \in R[x^{-1}]$, where $n = \deg f$. The two elements $f(x), g(x^{-1})$ are associates in $R[x, x^{-1}]$, so

$$V(f) \cap \operatorname{Spec} R[x^{-1}] = V(f) \cap \operatorname{Spec} R[x, x^{-1}] = V(g) \cap \operatorname{Spec} R[x, x^{-1}]$$

Since f is monic in x, we have $g \in 1 + x^{-1} \dot{R}[x^{-1}]$, so $V(g) \subset \operatorname{Spec} R[x, x^{-1}]$. Thus $V(f) \cap \operatorname{Spec} R[x^{-1}] = V(g)$ is closed in $\operatorname{Spec} R[x^{-1}]$, proving the claim.

Remark

This proof works over any commutative ring R as it does not invoke the hypothesis that R is local. The argument is due to Murthy.

Geometric proof of Local Horrock's Theorem

The main argument of the proof is to reduce the given bundle $\mathcal{E}$ on $\mathbb{P}^1_R$ modulo the maximal ideal, and then use certain facts from sheaf cohomology to pull back the information. We now state the results we use from the cohomology to formulate the proof.

Theorem (Grothendieck's Classification of Vector bundles)

If $\mathcal{E}$ is a vector bundle on $\mathbb{P}^1_k$, it is of the form $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$, with $a_1 \geq \cdots \geq a_r$ uniquely determined.

Theorem (Semicontinuity Theorem)

Let $\pi : X \to Y$ be a proper morphism of Noetherian schemes. Let $\mathcal{F}$ be a coherent sheaf on X flat over Y. Then the following holds:

1. *For each p, the function $y \mapsto dim_{\kappa(y)} H^p(X_y, \mathcal{F}_y)$ is upper semicontinuous on Y.*
2. *The function $y \mapsto \chi(\mathcal{F}_y) = \sum_p (-1)^p dim_{\kappa(y)} H^p(X_y, \mathcal{F}_y)$ is locally constant on Y.*

Theorem (The base change theorem)

Let Y be a Noetherian scheme, $\pi : X \to Y$ be a proper morphism, and $\mathcal{F}$ a coherent sheaf on X, flat over Y. Suppose $H^p(X_y, \mathcal{F} \otimes \kappa(y)) = 0$ for all $y \in Y$ and for some integer p. Then, if $\varphi : Y^{'} \to Y$ is any morphism and $\varphi^{'} : Y^{'} \times_Y X \to Y^{'}$ the base change, so that we have a cartesian square

$$\begin{array}{ccc} Y^{'} \times_Y X & \xrightarrow{\varphi^{'}} & X \\ \pi^{'} \big\downarrow & & \big\downarrow \pi \\ Y^{'} & \xrightarrow{\varphi} & Y \end{array}$$

then,

$$\varphi^* R^{p-1} \pi_*(\mathcal{F}) \cong R^{p-1} \pi^{'}_*((\varphi^{'})^*(\mathcal{F}))$$

Corollary

Let Y be a Noetherian scheme, $\pi : X \to Y$ be a proper morphism, and $\mathcal{F}$ a coherent sheaf on X, flat over Y. Suppose $H^1(X_y, \mathcal{F} \otimes \kappa(y)) = 0$ for all $y \in Y$, then

$$\pi_*(\mathcal{F}) \otimes \kappa(y) \cong H^0(X_y, \mathcal{F} \otimes \kappa(y))$$

Proof: Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^1_R$ which is extended from $\mathbb{A}^1_R$. Let us restrict $\mathcal{E}$ to the special fibre, Spec$k$ where $k = R/\mathfrak{m}$, the residue field, which we call $\mathcal{E}^{'}$. Then since k is a field, we can appeal to the theorem of Grothendieck and we see that

$$\mathcal{E}^{'} = \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_i),$$

for suitable integers $a_i^{\prime}s$. We twist $\mathcal{E}^{'}$ and correspondingly $\mathcal{E}$ by a suitable $\mathcal{O}(a)$, so that all the $a_i \geq 0$ and at least one $a_i = 0$. This ensures that our twisted $\mathcal{E}^{'}$ has nowhere vanishing section and also $H^1(\mathbb{P}^1_k, \mathcal{E}^{'}) = 0$. We have the following cartesian diagram

$$\begin{array}{ccc} \mathbb{P}^1_k & \longrightarrow & \mathbb{P}^1_R \\ \downarrow & & \downarrow \\ \mathrm{Spec}k & \longrightarrow & \mathrm{Spec}R \end{array}$$

If we show that $\mathcal{E}$ has a nowhere vanishing section then we have the following short exact sequence of locally free sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^1_R} \to \mathcal{E} \to \mathcal{E}/\mathcal{O}_{\mathbb{P}^1_R} \to 0$$

This will essentially say that restriction of $\mathcal{E}$ to the affine patch splits. In particular, let P be the projective module that comes from the restriction of $\mathcal{E}$ then $P = Q \oplus R[t]$ with rank of Q strictly less than the rank of P, then by induction on the rank of P finishes the proof.

We now give the proof of the existence of the nowhere vanishing section on $\mathcal{E}$.
Since R is local every closed subset of Spec$R$ contains unique maximal ideal $\mathfrak{m}$. Now, using semicontinuity we can say that

$$dim_{\kappa(y)} H^0(\mathbb{P}^1_{\kappa(y)}, \mathcal{E}_y) \leq dim_k H^0(\mathbb{P}^1_k, \mathcal{E}')$$

Moreover, R being local implies that the Euler characteristic $\chi(\mathcal{E}_y)$ is constant, from this we can conclude

$$dim_k H^0(\mathbb{P}^1_k, \mathcal{E}') \leq dim_{\kappa(y)} H^0(\mathbb{P}^1_{\kappa(y)}, \mathcal{E}_y)$$

Combining, we have $dim_k H^0(\mathbb{P}^1_k, \mathcal{E}') = dim_{\kappa(y)} H^0(\mathbb{P}^1_{\kappa(y)}, \mathcal{E}_y)$ and

$$H^1(\mathbb{P}^1_{\kappa(y)}, \mathcal{E}_y) = 0$$

for all $y \in$ Spec$R$.

Finally, if we say that the pullback map

$$H^0(\mathbb{P}^1_R, \mathcal{E}) \to H^0(\mathbb{P}^1_k, \mathcal{E}')$$

is surjective then we are done. But this map factors

$$\begin{array}{ccc} H^0(\mathbb{P}^1_R, \mathcal{E}) & \longrightarrow & H^0(\mathbb{P}^1_k, \mathcal{E}') \\ \downarrow & \nearrow_{\cong} & \\ \pi_* \mathcal{E} \otimes k & & \end{array}$$

where the isomorphism is given by the Base-change theorem as $H^1(\mathbb{P}^1_{\kappa(y)}, \mathcal{E}_y) = 0$ for $y \in \mathrm{Spec} R$. Hence this map is surjective and we are through.

Quillen-Suslin

Definition: We use the notation $\mathcal{M}(R)$ for the set of all finitely generated modules over a ring R and $\mathcal{P}(R)$ for the projective modules therein. If A is an R-algebra and $M \in \mathcal{M}(R)$ then we say that $A \otimes_R M \in \mathcal{M}(A)$ is extended from M and R and we write $Q \in \mathcal{M}^R(A)$ for all $Q \in \mathcal{M}(A)$ which are extended from R, and $\mathcal{P}^R(A)$ likewise.

Theorem (Quillen Patching)

Let R be any commutative ring, A is an R-algebra and $M \in \mathcal{M}(A[x_1, \dots, x_n])$ finitely presented. Then the following statements are equivalent

(A_n) $Q(M) := \{g \in R | M_g \in M^{A_g}(A_g[x_1 \dots x_n])\}$ *is an ideal of R*

(B_n) $(\forall \mathfrak{m} \in Max(R) : M_\mathfrak{m} \in M^{A_\mathfrak{m}}(A_\mathfrak{m}[x_1, \dots, x_n])) \Rightarrow M \in \mathcal{M}^A(A[x_1, \dots, x_n])$

Corollary

$P \in \mathcal{P}(R[x_1, \ldots, x_n])$ is extended from R iff $\forall \mathfrak{m} \in Max(R)$: $P_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}[x_1, \ldots, x_n]$-module.

Geometrically, this means an algebraic vector bundle on $\mathbb{A}^n \times X$ is extended from $X = \text{Spec} R$ iff this is the case for the neighbourhood of each closed point of X.

Fact. If R is a principal ideal domain, then $R\langle x\rangle$ is a principal ideal domain.

Theorem (Affine Horrocks)

Let R be any commutative ring and $P \in R[t]$. If $P\langle x\rangle = R\langle x\rangle \otimes_{R[x]} P \in \mathcal{P}^R(R\langle x\rangle)$, then $P \in \mathcal{P}^R(R[x])$.

Proof: Let $P \in \mathcal{P}(R[x])$ with $P\langle x\rangle \in \mathcal{P}^R(R\langle x\rangle)$. For $\mathfrak{m} \in \mathrm{Max}(R)$, $P\langle x\rangle_\mathfrak{m} \in \mathcal{P}^{R_\mathfrak{m}}(R\langle x\rangle_\mathfrak{m})$ and that implies $P\langle x\rangle_\mathfrak{m}$ is $R\langle x\rangle_\mathfrak{m}$ -free. By Local Horrocks theorem for $R_\mathfrak{m}$, $P_\mathfrak{m}$ is $R_\mathfrak{m}[x]$ free. By Quillen Patching, P is extended from R.

Proof of the Quillen Suslin Theorem

Proof: We proceed by induction on n, the base case $n = 0$ is trivial. Let $A := R[x_2, \ldots, x_n]$ and consider $A[x_1] \subset R\langle x_1\rangle[x_2, \ldots, x_n] \subset A\langle x_1\rangle$. If $P \in \mathcal{P}(R[x_1, \ldots, x_n])$, then $P \otimes_{R[x_1,\ldots,x_n]} R\langle x_1\rangle[x_2, \ldots, x_n]$ is a finitely generated $R\langle x_1\rangle[x_2, \ldots, x_n]$-module so by the induction hypotheses a free one. Hence, $P \otimes_{A[x_1]} A\langle x_1\rangle$ is a free $A\langle x_1\rangle$-module. Affine Horrocks implies that P is extended from $P/x_1P \in \mathcal{P}(A)$. Again by the induction hypothesis P/x_1P is A-free, so that P is $A[x_1]$-free.

A Question!

Suppose P be a projective $R[x]$- module where R is a Noetherian local ring and P also extends to a vector bundle on $\mathbb{P}^1_R$. Moreover, $P = Q \oplus R[x]$. Can we say that Q as a projective module can be extended to a vector bundle on $\mathbb{P}^1_R$?
This amounts to asking that every stably free module over $R[x]$ where R is Noetherian local ring is free. This was asked by Suslin and partially answered by Ravi Rao. This is open in general.