On the Goldbach Conjecture

Habibur Rahaman

Department of Mathematics and Statistics
Indian Institute of Science Education and Research Kolkata

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Chen's Theorem

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Every sufficiently large even integer can be written as sum of a prime and a number which is product of at most two primes.

The Twin Prime Counting Function

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$$\pi_2(x) \sim \frac{cx}{(\log x)^2} =: \Pi(x)$$

$$c := 2 \prod_{p>2} \frac{(1 - 2/p)}{(1 - 1/p)^2}.$$

The Twin Prime Counting Function

Year	Author(s)	$\pi_2(x)/\Pi(x) \lesssim$
1947	Selberg	8
1964	Pan	6
1966	Bombieri-Davenport	4
1978	Chen	3.9171
1983	Fouvry-Iwaniec	3.7777
1984	Fouvry	3.7647
1986	Bombieri-Friedander-Iwaniec	3.5
1986	Fouvry-Grupp	3.454
1990	Wu	3.418
2003	Cai-Lu	3.406
2004	Wu	3.399951
2021	Lichtman	3.29956

Chen’s Theorem

Chen's Theorem

Theorem.

Let $R(N)$ denote the number of representations of N as $N = p + n$, where p is a prime and n is product of at most two primes. Then for sufficiently large even N,

$$R(N) \gg \mathfrak{S}(N) \frac{2N}{(\log N)^2},$$

where $\mathfrak{S}(N) = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|N \\ p>2}} \frac{p-1}{p-2}.$

Sieving Function

- Let $A = \{a(n)\}_{n\geq 1}$be an arithmetic function, where $a(n) \geq 0$ and $|A| = \sum_n a(n) < \infty$.
- Let $\mathcal{P}$ be a set of primes and $z \geq 2$ be a real number.
- $P(z) := \prod_{\substack{p\in\mathcal{P} \\ p<z}} p.$

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- $P(z) := \prod_{\substack{p\in\mathcal{P} \\ p<z}} p.$
- $S(A, \mathcal{P}, z) := \sum_{(n,P(z))=1} a(n).$

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- Using above, $S(A, \mathcal{P}, z) = \sum_{d|P(z)} \mu(d)|A_d|$, where $|A_d| = \sum_{d|n} a(n)$.
- Let $g_n(d)$ multiplicative function with $0 \leq g_n(p) < 1$ for $p \in \mathcal{P}$.
- Remainder $r(d) := |A_d| - \sum_n a(n) g_n(d)$.

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- Using $r(d)$, we have $S(A, \mathcal{P}, z) = \sum_n a(n) V_n(z) + R(z)$, where $V_n(z) = \prod_{p|P(z)} (1 - g_n(p))$ and $R(z) = \sum_{d|P(z)} \mu(d) r(d)$

Sieve weights

- Let $D > 0$. Let $\lambda^+(d), \lambda^-(d)$ be arithmetic functions with
  $\lambda^+(1) = 1, \lambda^+(d) = 0$ for $d \geq D$, $\sum_{d|n} \lambda^+(d) \geq 0$ for $n \geq 2$
  $\lambda^-(1) = 1, \lambda^-(d) = 0$ for $d \geq D$, $\sum_{d|n} \lambda^-(d) \leq 0$ for $n \geq 2$.

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  $\lambda^-(1) = 1, \lambda^-(d) = 0$ for $d \geq D$, $\sum_{d|n} \lambda^-(d) \leq 0$ for $n \geq 2$.
- Let $F(s), f(s)$ are some functions satisfying

$$sF(s) = 2e^{\gamma}, \quad [1 \leq s \leq 3], \qquad (sF(s))' = f(s-1), \quad [s > 3],$$
$$sf(s) = 2e^{\gamma}\log(s-1), \quad [2 \leq s \leq 4], \qquad (sf(s))' = F(s-1), \quad [s > 2].$$

Jurkat-Richert Theorem

Let $\mathcal{Q}$ be a finite subset of $\mathcal{P}$ and for all n, $\displaystyle\prod_{\substack{p\in\mathcal{P}\setminus\mathcal{Q}\\ u\le p<z}}(1-g_n(p))^{-1}<(1+\epsilon)\frac{\log z}{\log u}$,

for some $\epsilon\in(0,1/200)$ and for any $1<u<z$.

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Jurkat-Richert Theorem.

For any $D \geq z$, there is upper bound as

$$S(A, \mathcal{P}, z) < (F(s) + \epsilon e^{14-s})X + R,$$

and for any $D \geq z^2$, there is a lower bound as

$$S(A, \mathcal{P}, z) > (f(s) - \epsilon e^{14-s})X - R,$$

where $s = \frac{\log D}{\log z}, X = \displaystyle\sum_n a(n)V_n(z)$ and the remainder term $R = \displaystyle\sum_{\substack{d|P(z)\\ d<DQ}} |r(d)|$.

Outline of the proof of Chen's Theorem

- Let $N \geq 4^8$ even integer, $z = N^{1/8}, y = N^{1/3}$.
- Let $\mathcal{P} = \{p < N : p \nmid N\}$.

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- Let $\mathcal{P} = \{p < N : p \nmid N\}$.
- $w(n) = 1 - \dfrac{1}{2} \sum\limits_{\substack{z \leq q < y \\ q^k || n}} k - \dfrac{1}{2} \sum\limits_{\substack{p_1 p_2 p_3 = n \\ z \leq p_1 < y \leq p_2 \leq p_3}} 1.$

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- For $n < N$ and $(n, N) = (n, P(z)) = 1$,
  $w(n) > 0 \Rightarrow n \in \{1, p_1, p_1 p_2 : p_1, p_2 \geq z\}$.
- Let $\mathcal{A} = \{N - p : p \leq N, p \in \mathcal{P}\}$.

Proof (continued.)

$$
\begin{aligned}
R(N) &\geq \sum_{\substack{n=N-p \\ n\in\{1,p_1,p_1p_2:p_1,p_2\geq z\}}} 1 \\
&\geq \sum_{\substack{n\in\mathcal{A} \\ (n,P(z))=1 \\ n\in\{1,p_1,p_1p_2:p_1,p_2\geq z\}}} w(n) \\
&\geq \sum_{\substack{n\in\mathcal{A} \\ (n,P(z))=1}} w(n) \\
&= \sum_{\substack{n\in\mathcal{A} \\ (n,P(z))=1}} 1 - \frac{1}{2} \sum_{\substack{n\in\mathcal{A} \\ (n,P(z))=1}} \sum_{\substack{z\leq q<y \\ q^k||n}} k - \frac{1}{2} \sum_{\substack{n\in\mathcal{A} \\ (n,P(z))=1}} \sum_{\substack{p_1p_2p_3=n \\ z\leq p_1<y\leq p_2\leq p_3}} 1. \qquad (1)
\end{aligned}
$$

Proof (continued.)

- Let $A = \{a(n)\}_{n=1}^{\infty}$ be characteristic function on $\mathcal{A}$.
- $A_q = \{a_q(n)\}_{n=1}^{\infty}$ be arithmetic function defined as

$$a_q(n) = \begin{cases} 1 & \text{if } n \in \mathcal{A} \text{ and } q|n, \\ 0 & \text{otherwise.} \end{cases}$$

Using above, first term of (1) reduces to $S(A, \mathcal{P}, z)$ and second term reduces to $\frac{1}{2} \sum_{z \le q < y} S(A_q, \mathcal{P}, z) + O(N^{7/8})$.

Proof (continued.)

Switching Principle

To compute the third term of (1), let us define the switched set

$$\mathcal{B} := \{N - p_1p_2p_3 : z \leq p_1 < y \leq p_2 \leq p_3, p_1p_2p_3 < N, (p_1p_2p_3, N) = 1\}.$$

Studying the third term of (1) is equivalent to bounding number of primes in $\mathcal{B}$.

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- Let $B = \{b(n)\}_{n=1}^{\infty}$ be characteristic function on $\mathcal{B}$.
- Then
$$\frac{1}{2} \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1}} \sum_{\substack{p_1p_2p_3=n \\ z \le p_1 < y \le p_2 \le p_3}} 1 = \frac{1}{2} S(B, \mathcal{P}, y) + O(N^{1/3}).$$

Proof (continued.)

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$$\frac{1}{2} \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1}} \sum_{\substack{p_1p_2p_3 = n \\ z \le p_1 < y \le p_2 \le p_3}} 1 = \frac{1}{2} S(B, \mathcal{P}, y) + O(N^{1/3}).$$

$$R(N) > S(A, \mathcal{P}, z) - \frac{1}{2} \sum_{z \le q < y} S(A_q, \mathcal{P}, z) - \frac{1}{2} S(B, \mathcal{P}, y) - 2N^{7/8} - N^{1/3}.$$

End of the Proof

Using Jurkat-Richert Theorem and Bombieri-Vinogradov Theorem, we have the following bounds.

- $S(A, \mathcal{P}, z) > \left(\frac{e^{\gamma}\log 3}{2} + O(\epsilon)\right)\frac{NV(z)}{\log N}$.
- $\displaystyle\sum_{z \le q < y} S(A_q, \mathcal{P}, z) < \left(\frac{e^{\gamma}\log 6}{2} + O(\epsilon)\right)\frac{NV(z)}{\log N}$.
- $S(B, \mathcal{P}, y) < \left(\frac{ce^{\gamma}}{2} + O(\epsilon)\right)\frac{NV(z)}{\log N} + O\left(\frac{N}{\epsilon(\log N)^3}\right)$.

Using Mertens's formula, we have,

- $V(z) = \mathfrak{S}(N)\dfrac{e^{-\gamma}}{\log z}\left(1 + O(\dfrac{1}{\log N}\right)$.

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- $\displaystyle\sum_{z \le q < y} S(A_q, \mathcal{P}, z) < \left(\frac{e^\gamma \log 6}{2} + O(\epsilon)\right) \frac{NV(z)}{\log N}$.
- $S(B, \mathcal{P}, y) < \left(\frac{ce^\gamma}{2} + O(\epsilon)\right) \frac{NV(z)}{\log N} + O\left(\frac{N}{\epsilon(\log N)^3}\right)$.

Using Mertens's formula, we have,

- $V(z) = \mathfrak{S}(N) \dfrac{e^{-\gamma}}{\log z} \left(1 + O(\dfrac{1}{\log N}\right)$.

Putting all these bounds in the expression of $R(N)$, we have,

$$R(N) \gg \mathfrak{S}(N) \frac{2N}{(\log N)^2}.$$

References

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Thank you!