Representation of $\mathfrak{sl}(2, \mathbb{C})$

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Definition of Lie algebra

Lie Algebra:- A finite dimensional real or complex lie algebra is a finite dimensional complex or real vector space $\mathfrak{g}$, together with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that the following properties holds ;

1. $[\cdot,\cdot]$ is bilinear.
2. $[\cdot,\cdot]$ is skew symmetric: $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$
3. The **Jacobi identity** holds;

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

Lie Subalgebra:- A subalgebra of a real or complex lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[H_1, H_2] \in \mathfrak{h}$ for all $H_1, H_2 \in \mathfrak{h}$.

Ideal:- A subalgebra $\mathfrak{h}$ of a lie algebra $\mathfrak{g}$ is said to be **Ideal** in $\mathfrak{g}$ if $[X, H] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$.

A Lie algebra $\mathfrak{g}$ is called **irreducible** if the only ideals in $\mathfrak{g}$ are $\mathfrak{g}$ and $\{0\}$.A Lie algebra $\mathfrak{g}$ is called **Simple** if it is irreducible and $\dim\mathfrak{g} \geq 2$.

$\mathfrak{sl}(2, \mathbb{C})$ and its properties

$\mathfrak{sl}(2, \mathbb{C}) = \{X \in M_2(\mathbb{C}) | \, trace(X) = 0\}$, the bracket operation is given by

$$[X, Y] = XY - YX$$

Then $\mathfrak{sl}(2, \mathbb{C})$ is a lie complex lie algebra with respect to this bracket operation.

Some basic properties of $\mathfrak{sl}(2, \mathbb{C})$:-

1. dim($\mathfrak{sl}(2, \mathbb{C})$)=3.
2. $\mathfrak{sl}(2, \mathbb{C})$ is simple.
3. $\mathfrak{sl}(2, \mathbb{C})$ is the isomorphic to the complexification of real lie algebra $\mathfrak{su}(2)$ which comes from the lie group $SU(2)$.
4. $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a basis of $\mathfrak{sl}(2, \mathbb{C})$.
5. $[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$.

Representation of Lie algebra

Definition:- If $\mathfrak{g}$ is a real or complex lie algebra, then a **finite dimensional complex representation** of $\mathfrak{g}$ is a lie algebra homomorphism π of $\mathfrak{g}$ into $\mathfrak{gl}(V)$, where V is a finite dimensional complex vector space.

The notion of invariant subspace, irreducible representation, intertwining map is same as of group representations.

Important results

Proposition

Let G be a matrix lie group with lie algebra $\mathfrak{g}$ and let $\prod$ be a(finite dimensional real or complex) representation of G,acting on space V.Then there is a unique representation π of $\mathfrak{g}$ acting on the same space V such that for all $X \in \mathfrak{g}$

$$\prod(e^X) = e^{\pi(X)} \textit{ and } \pi(X) = \frac{d}{dt}(\prod(e^{tX}))\Big|_{t=0}$$

Two important results

Proposition

Let G be a connected matrix Lie group with lie algebra $\mathfrak{g}$.Let $\prod$ be a representation of G and π be the associated representation of $\mathfrak{g}$.Then $\prod$ is irreducible if and only if π is irreducible.

Proposition

Let $\mathfrak{g}$ be real lie algebra and $\mathfrak{g}_{\mathbb{C}}$ be its complexification.Then every finite dimensional complex representation π of $\mathfrak{g}$ has a unique extension to complex linear representation of $\mathfrak{g}$,also denoted π.Furthermore,π is irreducible as a representation of $\mathfrak{g}_{\mathbb{C}}$ if and only if it is irreducible as a representation of $\mathfrak{g}$.

Let V_m be space of homogenous polynomial of degree m in two complex variable,that means elements of V_m look like,

$$f(z_1, z_2) = a_0 z_1^m + a_1 z_1^{m-1} z_2 + a_2 z_1^{m-2} z_2^2 + \ldots + a_m z_2^m$$

Where a_i are complex numbers.Clearly dim(V_m)=m+1.
Now let us define the action of $SU(2)$ on V_m.For each $U \in SU(2)$ define a linear transformation $\Pi_m(U)$ on the space V_m by the formula,

$$[\Pi_m(U)f]z = f(U^{-1}z) \text{ for } z \in \mathbb{C}^2$$

This is indeed a representation of SU(2) that is we need to show $\Pi_m(U_1)(\Pi_m(U_2)) = \Pi_m(U_1U_2)$ now let $f \in V_m$ and $z \in \mathbb{C}^2$,then

$$\begin{aligned}\Pi_m(U_1)(\Pi_m(U_2)f)z &= [\Pi_m(U_2)f](U_1^{-1}z)\\ &= f(U_2^{-1}U_1^{-1}z)\\ &= [\Pi_m(U_1U_2)f]z\end{aligned}$$

Hence Π_m is indeed a m+1 dimensional representation of $SU(2)$

Induced representation π_m of $\mathfrak{su}(2)$ through $SU(2)$

Now the associated representation π_m of $\mathfrak{su}(2)$ is given by

$$\pi_m(X) = \frac{d}{dt}\Pi_m(e^{tX})\Big|_{t=0}$$

Now $\pi_m(X)$ is a map from V_m to V_m,thus

$$(\pi_m(X)f)z = \frac{d}{dt}(\Pi_m(e^{tX})f)z\Big|_{t=0} = \frac{d}{dt}f(e^{-tX}z)\Big|_{t=0}$$

Here calculation is bit long so i will skip this calculation and state the important results,

Proposition

For each $m \geq 0$,the representation π_m is irreducible.

Representation of $\mathfrak{sl}(2,\mathbb{C})$

Now first let fix the basis of $\mathfrak{sl}(2,\mathbb{C})$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which have commutation relations, $[H,X] = 2X, [H,Y] = -2Y, [X,Y] = H$. **Basic idea**:- Now previously while dealing with representation $SU(2)$ and its lie algebra $\mathfrak{su}(2)$ for each $m \geq 0$ there is a irreducible representation π_m of $\mathfrak{sl}(2,C)$ having dimension m+1.Now what we will show now that this are the only irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$,here is the main result

Theorem

For each $m \geq 0$,there is an irreducible complex representation of $\mathfrak{sl}(2,\mathbb{C})$ with dimension m+1.Any two irreducible complex representations of $\mathfrak{sl}(2,\mathbb{C})$ with same dimension are isomorphic.If π is an irreducible complex representtion of $\mathfrak{sl}(2,\mathbb{C})$ with dimension m+1,then π is isomorphic to representation π_m.

Lemma

Let u be an eigenvector of $\pi(H)$ *with an eigenvalue* $\alpha \in \mathbb{C}$. *Then we have*

$$\pi(H)\pi(X)u = (\alpha + 2)\pi(X)u$$

Thus either $\pi(X)u = 0$ *or* $\pi(X)u$ *is an eigenvector for* $\pi(H)$ *with eigenvalue* $\alpha + 2$. *Similarly,*

$$\pi(H)\pi(Y)u = (\alpha - 2)\pi(Y)u$$

Thus either $\pi(Y)u = 0$ *or* $\pi(Y)u$ *is an eigenvector for* $\pi(H)$ *with eigenvalue* $\alpha - 2$

Proof of previous lemma

Proof.

Now π is a Lie algebra homomorphism then,$\pi([H, X]) = [\pi(H), \pi(X)]$ and hence

$$\begin{aligned} \pi(2X) &= \pi(H)\pi(X) - \pi(X)\pi(H) \\ 2\pi(X)(u) &= \pi(H)(\pi(X)u) - \pi(X)(\pi(H)u) \\ &= \pi(H)(\pi(X)u) - \alpha\pi(X)u \end{aligned}$$

And thus we get the desired result.Similar calculation we can do on $[H, Y]$ to get second result. □

Classifying the irrep representation of $\mathfrak{sl}(2, \mathbb{C})$

Now the basic strategy is to diagonalize $\pi(H)$ in some nice way.Since we are working over finite dimensional complex vector space there must be an eigenvector of $\pi(H)$ say u corresponding to eigenvalue $\alpha \in \mathbb{C}$.Now if we apply repeatedly the previous lemma,we get

$$\pi(H)(\pi(X)^k u) = (\alpha + 2k)(\pi(X)^k u), k \geq 0$$

Since α is fixed and so for two different k_1 and k_2, $\alpha + 2k_1 \neq \alpha + 2k_2$.Now if all of $\pi(X)^k u \neq 0$ then we will get infinitely many linearly independent vectors which is not possible in finite dimensional vector space.Thus there is some $N \geq 0$ such that

$$\pi(X)^N u \neq 0$$

But,

$$\pi(X)^{N+1} u = 0$$

Now let's say, $u_0 = \pi(X)^N u$ and $\lambda = \alpha + 2N$,then,

$$\begin{aligned} \pi(H)u_0 &= \lambda u_0 \\ \pi(X)u_0 &= 0 \end{aligned} \tag{1}$$

Continue the proof

Now let us define

$$u_k = \pi(Y)^k u_0$$

for $k \geq 0$.Now u_0 is an eigenvector for $\pi(H)$ corresponding to eigenvalue λ.Similarly if we apply lemma repeatedly for this eigenvector we get,

$$\pi(H)u_k = (\lambda - 2k)u_k$$

Now we will see the action of u_k on $\pi(X)$,for $k = 1$ by using the fact that $\pi(X)u_0 = 0$ and commutator relation,

$$\begin{aligned}\pi(X)u_1 &= \pi(X)(\pi(Y)u_0) \\ &= \pi(H)u_0 + \pi(Y)(\pi(X)u_0) \\ &= \lambda u_0\end{aligned}$$

In genral,

$$\pi(X)u_k = k(\lambda - (k-1))u_{k-1}, k \geq 1 \tag{2}$$

Let's do some more calculation

Now let us prove the equation 2 by induction,first observe that
$u_{k+1} = \pi(Y)^{k+1} u_0 = \pi(Y)(\pi(Y)^k u_0) = \pi(Y) u_k$
Now, by using the induction hypothesis and commutator relation we get

$$\begin{aligned}
\pi(X)(\pi(Y)u_k) &= \pi(H)u_k + \pi(Y)(\pi(X)u_k) \\
&= (\lambda - 2k)u_k + \pi(Y)([k(\lambda - (k-1))]u_{k-1}) \\
&= (\lambda - 2k)u_k + [k(\lambda - (k-1))]\pi(Y)(u_{k-1}) \\
&= (\lambda - 2k)u_k + [k(\lambda - (k-1))](u_k) \\
&= (k+1)(\lambda - k)u_k \\
&= (k+1)(\lambda - ((k+1) - 1)u_k \\
\pi(X)u_{k+1} &= (k+1)(\lambda - ((k+1) - 1)u_k
\end{aligned}$$

Hence we get,

$$\pi(X)u_k = k(\lambda - (k-1))u_{k-1}$$

Calculations,Calculations,Calculation.....!!!

Since, $\pi(H)u_k = (\lambda - 2k)u_k$ and $\pi(H)$ has only finitely many eigenvalues,all $u_k's$ cannot be non zero.So there must be a non-negative integer m such that

$$u_k = \pi(Y)^k u_0 \neq 0$$

for all $k \leq m$,but

$$u_{m+1} = \pi(Y)^{m+1} u_0 = 0$$

Now $u_{m+1} = 0$,then $\pi(X)u_{m+1} = 0$ which in turn means that $(m+1)(\lambda - m)u_m = 0$,since $u_m \neq 0$ and m+1 is non zero imply that $\lambda = m$.

All things in one box

Finally for every representation (π, V),**there exists an integer m $\geq$ 0** and non-zero vectors $u_0, u_1, \ldots, u_m$ such that

$$\pi(H)u_k = (m-2k)u_k$$
$$\pi(Y)u_k = \begin{cases} u_{k+1}, & \text{k} < \text{m} \\ 0, & \text{k=m} \end{cases}$$
$$\pi(X)u_k = \begin{cases} k(m-(k-1))u_{k-1}, & \text{k} > 0 \\ 0, & \text{k=0} \end{cases} \tag{3}$$

Remark

Note that till now we have not used irreducibility of π.

Almost near the shore,Really !!

Now we know that eigenvectors corresponding to distinct eigenvalues are linearly independent,thus $u_0, u_1 \dots u_m$ forms a linearly independent set in V.Let $W = Span\{u_0, u_1 \dots u_m\}$,then dim(W)=m+1 also each basis element of W under $\pi(H), \pi(X), \pi(Y)$ maps to some scalar multiple of basis hence can we conclude that $\pi(A)W \subset W$ for all $A \in \mathfrak{sl}(2, \mathbb{C})$.Now π is irreducible imply that W=V.

Finally if we take any irreducible representation we know how it will look like by equation 3.Conversely we can ask whether there exists such a irreducible representation.Not let us define the representation ρ on a m+1 dimensional complex vector space V' having basis $u_0, u_1, \dots, u_m$

$$\rho(H)u_k = (m-2k)u_k$$
$$\rho(Y)u_k = \begin{cases} u_{k+1}, & \text{k} < \text{m} \\ 0, & \text{k=m} \end{cases} \tag{4}$$
$$\rho(X)u_k = \begin{cases} k(m-(k-1))u_{k-1}, & \text{k} > 0 \\ 0, & \text{k=0} \end{cases}$$

ρ is an irreducible representation of $V^{'}$

Now we will show ρ is a lie algebra homomorphism for that first we will show bracket operation will preserved by [X,Y], [H,X] and [H.Y] under ρ ,Since H, X, Y be basis so we are done.Now, $\rho([X,Y]) = [\rho(X), \rho(Y)]$ which in turn imply $\rho(H) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$.Now consider three different cases;

Case 1 Acting on $\mathbf{u_0}$:-Then $\rho(H)u_0 = mu_0$ and $(\rho(X)\rho(Y) - \rho(Y)\rho(X))u_0 = \rho(X)(u_1) = mu_0$ and hence LHS=RHS.

Case 2 Acting on $\mathbf{u_m}$:-Then $\rho(H)u_m = -mu_m$ and $(\rho(X)\rho(Y) - \rho(Y)\rho(X))u_m = -\rho(Y)(mu_{m-1}) = -m\rho(Y)(u_m) = -mu_m$.Again LHS=RHS.

Case 3 Acting on $\mathbf{u_k, 0 < k < m}$:- Now $\rho(H)u_k = (m-2k)u_k$ and $(\rho(X)\rho(Y) - \rho(Y)\rho(X))u_k = \rho(X)(u_{k+1}) - \rho(Y)(k(m-(k-1))u_{k-1}) = (k+1)(m-k)u_k - (k(m-(k-1))\rho(Y)(u_k) = (mk - k^2 + m - k - k(m-k+1))u_k = (mk - k^2 + m - k - mk + k^2 - k)u_k = (m-2k)u_k$ Again LHS=RHS.

Final discussion

Hence $\rho([X, Y]) = [\rho(X), \rho(Y)]$ for X, Y in V' since both of them are linear transformation and agree on the basis elements.**Similarly we can verify for [H,X] and [H,Y]** and then for all $A = a_1 H + a_2 X + a_3 Y \in \mathfrak{sl}(2, \mathbb{C})$.

ρ **is irreducible**:-Let $w = a_0 u_0 + a_1 u_1 + \ldots + a_m u_m$ be non zero element in a non-zero vector ρ-invariant subspace W of V' let k_0 be the least positive integer such that $a_{k_0} \neq 0$.Now observer that $\rho(X)^j(u_k) = 0$ for all $j > k$.Thus $\rho(X)^j$(w) is a non-zero multiple of u_0 for sufficiently large j but not to large, and hence W contains u_0,Now apply $\rho(Y)$ on u_0 we get all the basis elements in W.Thus V=W and consequently V is irreducible. Now by Above calculation it is clear any two irreducible representation of dimension m+1 will look like equations 3 and hence isomorphic and hence upto isomorphism there is only one irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ and we have one such representation π_m which is constructed from $SU(2)$.Thus $\pi \cong \pi_m$.

Why the representation of $\mathfrak{sl}(2,\mathbb{C})$?

1. To study the Representation of $\mathfrak{sl}(3,\mathbb{C})$,representation of $\mathfrak{sl}(2,\mathbb{C})$ is quite important since $\mathfrak{sl}(3,\mathbb{C})$ contains 2 copies of $\mathfrak{sl}(2,\mathbb{C})$.As the basis of $\mathfrak{sl}(3,\mathbb{C})$ is given by,

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

,Now Span(H_1, X_1, Y_1) and Span(H_2, X_2, Y_2) is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

One to One correspondence between representation of matrix Lie group and Lie algebra

If $\Pi : G \rightarrow GL(V)$ is a finite dimensional irreducible representation of G.Then by the main theorem the corresponding representation of Lie algebra $\mathfrak{g}$ of a matrix lie group is also irreducible.Now we ask **Whether every irreducible representation of Lie algebra $\mathfrak{g}$ comes from matrix lie group**(here lie algebra comes from some matrix lie group) In general this is not the case.

Let's Observe the representation of $\mathfrak{su}(2)$ Now irreducible representation of $\mathfrak{su}(2)$ are in one to one correspondence with irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ and we have just proved that any irreducible representation of $\mathfrak{sl}(2, C)$ having dimension m+1 is isomorphic to π_m which in turn comes from the representation Π_m of $SU(2)$ and **Note that SU(2) is simply connected**.

Representation of $SO(3)$ and its associated lie algebra $\mathfrak{so}(3)$

Now consider a lie algebra isomorphism $\phi : \mathfrak{su}(2) \to \mathfrak{so}(3)$ which sends $E_1 \to F_1$,$E_2 \to F_2$ and $E_3 \to F_3$ where

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, E_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and,

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Now if σ is an irreducible representation of $\mathfrak{so}(3)$ then $\sigma \circ \phi$ is irrep of $\mathfrak{su}(2)$,thus $\sigma \circ \phi = \pi_m$ for unique m and hence $\sigma = \pi_m \circ \phi^{-1}$ and these are the only irreducible representation of $\mathfrak{so}(3)$.But we can see that for odd m the representation $\sigma_m = \sigma$ can never be come from representation of matrix lie group $SO(3)$.

Cheers to all, you are free now !!