On extension of Calderón-Zygmund Type Singular Integrals and their commutators

Joydwip Singh

Department of Mathematics and Statistics
IISER Kolkata

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IISER KOLKATA

Goal

In today's talk we first study classical Calderón-Zygmund singular integral. Then we define an extension of it. We will estimate this extension Calderón-Zygmund type singular integral on various spaces and from where we deduce boundedness of classical Calderón-Zygmund singular integral on those spaces. At last we will discuss about some open problems in this direction.

Preliminaries

- For $1 \leq p < \infty$ we say $f \in L^p(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n} |f(x)|^p dx < \infty$.
- For f, g a suitable class of functions (i.e. take $f, g \in L^1(\mathbb{R}^n)$), we define
$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$
- The Schwartz space or the space of rapidly decreasing functions, $\mathcal{S}$, is given by
$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) : \lim_{|x|\to\infty} |x^\beta D^\alpha f(x)| = 0, \quad \forall \alpha, \beta \in \mathbb{N}^n\}.$$

Example

1. $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.
2. $\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$.

Motivation to define singular integral

Suppose $f \in L^p(\mathbb{R})$ $(1 \leq p < \infty)$. Consider the Cauchy integral on $\mathbb{R}$:

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - z} dt,$$

where $z = x + iy$, $y > 0$. It is easy to see that $F(z)$ is analytic on $\mathbb{R}^2_+$. Note that

$$\begin{aligned} F(z) &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} f(t) dt + \frac{i}{2\pi} \int_{\mathbb{R}} \frac{x-t}{(x-t)^2 + y^2} f(t) dt \\ &:= \frac{1}{2} \left[(P_y * f)(x) + i (Q_y * f)(x) \right], \end{aligned}$$

where $P_y(t) = \frac{1}{\pi} \frac{y}{t^2+y^2}$ is called Poisson kernel and $Q_y(t) = \frac{1}{\pi} \frac{t}{t^2+y^2}$ is called conjugate Poisson kernel.

Approximation of identity

Note that $P_y(t) = \frac{1}{\pi}\frac{y}{t^2+y^2}$ has the following properties.

- $P_y(t) > 0 \ \ \forall t \in \mathbb{R}$ and $y \in \mathbb{R}^+$.
- $\int_{-\infty}^{\infty} P_y(t)dt = 1$.
- As $y \to 0$,
  $P_y(t) \to \delta$, Dirac measure at the origin in $\mathcal{S}'$.

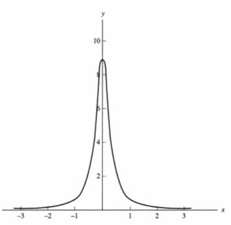

Figure: Poisson kernel

Then for $g \in \mathcal{S}$, we have the pointwise limit

$$\lim_{y\to 0} P_y * g(t) = g(t).$$

Because of this we say that $\{P_y\}$ is an approximation of the identity.

conjugate Poisson kernel

We would like to do the same for $Q_y(t) = \frac{1}{\pi}\frac{t}{t^2+y^2}$, but we immediately run into an obstacle: $\{Q_y\}$ is not an approximation of identity and, in fact, Q_y is not integrable for any $y > 0$. Formally,

$$\lim_{y\to 0} Q_y(t) = \frac{1}{\pi t}.$$

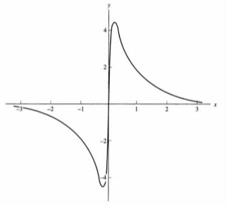

Figure: Conjugate Poisson kernel

principal value

- We define a 'tempered distribution' called the principal value of $1/x$, by

$$p.v.\frac{1}{x}(\phi) = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\phi(x)}{x}dx, \quad \phi\in\mathcal{S}.$$

- Then one can show that, $\lim_{y\to 0} Q_y(x) = \frac{1}{\pi}p.v.\frac{1}{x}$ in $\mathcal{S}'$.
- As a consequence we get

$$\lim_{y\to 0} Q_y * f(x) = \frac{1}{\pi}p.v.\frac{1}{x} * f = \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|t|>\epsilon}\frac{f(x-t)}{t}dt.$$

Hilbert transform

Given a function $f \in \mathcal{S}$, we define Hilbert transform on $\mathbb{R}$ by

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{1}{y} f(x-y) dy.$$

Note that we can write

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{sgn(y)}{|y|} f(x-y) dy.$$

Then we have

- $sgn(ry) = sgn(y)$ for $r > 0$ and $y \neq 0$.
  So it is enough to define it on $S^0 = \{1, -1\}$.
- $sgn(y') \in L^\infty(S^0)$.
- $\int_{S^0} sgn(y') d\sigma(y') = 0$.

Calderón-Zygmund (CZ) type singular integral

More generally for $f \in \mathcal{S}(\mathbb{R}^n)$, we can define Calderón-Zygmund type singular integral by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy,$$

where Ω is assumed to satisfy the following conditions:

$$\begin{cases} \Omega \in L^\infty(S^{n-1}), & (L^\infty\text{-bounded}) \\ \Omega(rx') = \Omega(x'), & (\text{homogeneous of degree } 0) \\ \int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0, & (\text{cancellation}), \end{cases} \tag{1}$$

for every $r > 0$, with S^{n-1} denoting the unit sphere $\{x' \in \mathbb{R}^n : |x'| = 1\}$.

Dini type condition

A function Ω is said to satisfy Dini type condition if

$$\int_0^1 \frac{\omega_\infty(\delta)}{\delta}\, d\delta < \infty, \tag{2}$$

where $\omega_\infty(\delta) := \sup\left\{\left|\Omega(x') - \Omega(y')\right| : \left|x' - y'\right| \leq \delta, |x'| = |y'| = 1\right\}.$

Example

- If we take $\Omega(x) = sgn(x)$ then it will be Hilbert transform.
- Take $\Omega(x) = x_j/|x|$ with $j = 1, 2, \ldots, n$, then it is called Riesz transform.

It is not difficult to verify that both the function satisfies the conditions (1) and (2).

Boundedness of Calderón-Zygmund operator

Theorem ([LDY07])

Let T be a Calderón-Zygmund operator and Ω satisfies condition (1) *and* (2) *then T is 'strong' (p, p) for $1 < p < \infty$ i.e.*

$$\|Tf\|_{L^p} \leq C\|f\|_{L^p}.$$

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Endpoint estimate

Calderón-Zygmund operator is not bounded in L^1 or L^∞, but one can show that $T : H^1 \to L^1$ and $L^\infty \to BMO$ are bounded.

Calderón-Zygmund commutator

Now consider Calderón-Zygmund commutator $[b, T]$ generated by a Calderón-Zygmund type singular integral T and a measurable function b on $\mathbb{R}^n$ is defined by

$$\begin{aligned} [b, T]f(x) &:= b(x)Tf(x) - T(bf)(x) \\ &= p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n}[b(x) - b(y)]f(y)\, dy, \end{aligned}$$

where $f \in \mathcal{S}$, and Ω satisfies (1).

BMO space

Consider the Fefferman–Stein sharp maximal function, $M^\sharp f$ of f given by

$$M^\sharp f(x) := \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x and $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$, which is called the average of f.

Definition

Define

$$BMO := \{f \in L^1_{loc}(\mathbb{R}^n) : M^\# f \in L^\infty\},$$

with the norm of $f \in BMO$ (upto a difference by constants) given by

$$\|f\|_{BMO} := \|M^\# f\|_{L^\infty}.$$

Homogeneous Lipschitz space

Example

- Clearly $L^\infty \subset BMO$.
- But there also unbounded BMO functions, i.e. $\log(|x|) \in BMO(\mathbb{R}^n)$.

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Definition

For $0 < \gamma < 1$, the homogeneous Lipschitz space $Lip_\gamma(\mathbb{R}^n)$ consists of functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{Lip_\gamma} := \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\gamma} < \infty.$$

Boundedness of commutator

Theorem ([LDY07])

Let $[b, T]$ be the Calderón-Zygmund commutator and Ω satisfies condition (1) and (2) then

1. $b \in BMO$ *iff* $[b, T]$ *is strong* (p, p) *for* $1 < p < \infty$.
2. $b \in Lip_{\gamma}(\mathbb{R}^n)$ *iff* $[b, T]$ *is strong* (p, q) *for* $1 < p < q < \infty$ *and* $1/q = 1/p - \gamma/n$.

Boundedness of commutator

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Endpoint estimate

1. For $b \in BMO$, $[b, T]$ is not (H^1, L^1), but it is 'weak' (H^1, L^1).
2. For $b \in Lip_\gamma$, $[b, T]$ is of (H^p, L^q) type if $n/(n+\gamma) < p \leq 1$ and $1/q = 1/p - \gamma/n$.

Extension of CZ type singular integrals and its commutators

Let $f \in \mathcal{S}$ and Ω as early.

- We consider the following extension of CZ type singular integrals defined, for $0 < \beta < n$, by

$$T_\beta f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\beta}} f(x-y)\, dy.$$

- We also consider the following extension of the commutator of the CZ type operator defined by,

$$\begin{aligned}[b, T_\beta]f(x) &:= b(x)\, T_\beta f(x) - T_\beta(bf)(x) \\ &= p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\beta}} [b(x) - b(y)] f(y)\, dy.\end{aligned}$$

Theorem ([LDY07])

Suppose $0 < \beta < n$ *and* Ω *satisfies all the hypothesis as above. Let* $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$ *and* $f \in L^p(\mathbb{R}^n)$ *for* $1 < p < \frac{n}{\beta}$ *,then*

$$\|T_\beta(f)\|_{L^q} \leq C_\beta \|f\|_{L^p}.$$

Boundedness of T_β and $[b, T_\beta]$

Theorem ([LDY07])

Suppose $0 < \beta < n$ *and* Ω *satisfies all the hypothesis as above. Let* $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$ *and* $f \in L^p(\mathbb{R}^n)$ *for* $1 < p < \frac{n}{\beta}$ *,then*

$$\|T_\beta(f)\|_{L^q} \leq C_\beta \|f\|_{L^p}.$$

Theorem ([LDY07])

1. *For* $b \in BMO$ *we have* $\|[b, T_\beta]f\|_{L^q} \leq C_\beta \|f\|_{L^p}$, *for* $0 < \beta < n$ *and* $1 < p < q < \infty$ *such that* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$.
2. *For* $b \in Lip_\gamma(\mathbb{R}^n)$ *we have* $\|[b, T_\beta]f\|_{L^q} \leq C_\beta \|b\|_{Lip_\gamma} \|f\|_{L^p}$, *for* $0 < \beta < n$, $0 < \gamma < 1$, *and* $1 < p < q < \infty$ *such that* $\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{n} + \frac{\beta}{n}$.

Problem

- Formally T_β and $[b, T_\beta]$ becomes CZ type singular integral and its commutator respectively when $\beta = 0$.
- Note that in the last theorem all the constant C_β depends on β and one can check that if we take $\beta \to 0$ the constant become unbounded. So we can not recover boundedness result of T from boundedness of T_β by taking $\beta \to 0$.
- So our problem is to find an uniform estimate of T_β and $[b, T_\beta]$ with respect to $\beta > 0$, such that strong boundedness of T and $[b, T]$ can be recovered when $\beta \to 0$.

Positive result for L^p

Recently [YJL21] proved the following boundedness result.

Theorem

Let $0 < \beta_0 < \frac{1}{2}$ *be any fixed and small number. Then for any* $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ *with* $1 < q < \infty$ *there exists constant* C *independent of* β *such that*

$$\|T_\beta f\|_{L^q} \leq C \left(\|f\|_{L^q} + \frac{\beta^{\frac{(q-1)n}{q}}}{\sqrt[q]{(n(q-1) - \beta q)}} \|f\|_{L^1} \right)$$

holds uniformly for $\varepsilon > 0$ *and* $0 < \beta < \min\left\{1 - \beta_0, \frac{(q-1)n}{q}\right\}$.

Atoms

Now we want to extend this result for $p \leq 1$. For $p \leq 1$, L^p spaces are not well behaved for boundedness of singular integral, the spaces that are much better suited to a host of question in harmonic analysis is H^p(Hardy spaces).

Definition

Let $0 < p \leq 1 \leq q \leq \infty$, $p < q$, and $s \geq \left[n(\frac{1}{p} - 1)\right]$. A (p, q, s)-atom centered at x_0 is a function $a \in L^q(\mathbb{R}^n)$ supported on a ball $B \subset \mathbb{R}^n$ with centre x_0 and satisfying,

1. $||a||_{L^q} \leq |B|^{\frac{1}{q} - \frac{1}{p}}$,
2. $\displaystyle\int_{\mathbb{R}^n} a(x)x^\alpha \, dx = 0$, for all $|\alpha| \leq s$.

Hardy spaces

Definition

Given $0 < p \leq 1$, the Hardy space H^p consists of all tempered distributions f admitting a decomposition $f = \sum_j \lambda_j a_j$, where a_j are (p, q, s)-atoms and $\sum_j |\lambda_j|^p < \infty$.

We also define $\|f\|_{H^p} := \inf \left(\sum_j |\lambda_j|^p \right)^{1/p}$, with the infimum being taken over all admissible representations $f = \sum_j \lambda_j a_j$.

Example

Let f be an odd function and $supp(f) \subset [-1, 1]$ with $|f(x)| \leq \frac{1}{2}$. Then $f \in H^1$.

Dini α-type condition

In some of our results, we shall also need the following stronger Dini type condition: for some $\alpha \in (0, 1]$,

$$\int_0^1 \frac{\omega_\infty(\delta)}{\delta^{1+\alpha}}\, d\delta < \infty \tag{3}$$

where $\omega_\infty(\delta) := \sup\left\{ \left|\Omega(x') - \Omega(y')\right| : \left|x' - y'\right| \le \delta, |x'| = |y'| = 1 \right\}.$

Example

One can show that Riesz kernel $\Omega(x) = x_j/|x|$ with $j = 1, 2, \ldots, n$, satisfies the condition (3).

Lipschitz space estimate

It is well known that the dual of the Hardy space H^p $(0 < p \leq 1)$ is Lipschitz space $Lip_{n(\frac{1}{p}-1)}$ whenever $\frac{n}{n+1} < p < 1$.

Theorem ([BGS22])

Let Ω satisfy conditions (1) and (3) for some $0 < \alpha \leq 1$. If $\frac{n}{n+\alpha} < p < 1 < q < \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 2$, then there exists a constant $C > 0$ such that

$$\|T_\beta f\|_{Lip_{n(\frac{1}{p}-1)}} \leq C \left(\|f\|_{Lip_{n(\frac{1}{p}-1)}} + \frac{\beta^{\frac{(q-1)n}{q}}}{\sqrt[q]{(n(q-1)-\beta q)}} \|f\|_{BMO} \right).$$

holds true for all $f \in Lip_{n(\frac{1}{p}-1)} \cap BMO$ and $0 < \beta < \frac{(q-1)n}{q}$.

Hardy space estimate

We have the following estimates of operators T_β on Hardy spaces H^p, with error term taken in other Hardy spaces.

Theorem ([BGS22])

Let Ω satisfy conditions (1) and (3) for some $0 < \alpha \leq 1$. Given $\frac{n}{n+\alpha} < m < p \leq 1$, let $q > 1$ be such that $\frac{1}{m} - \frac{1}{p} = 1 - \frac{1}{q}$. Then there exists a constant $C > 0$ such that

$$\|T_\beta f\|_{H^p} \leq C \left(\|f\|_{H^p} + \frac{\beta^{\frac{(q-1)n}{q}}}{\sqrt[q]{(n(q-1) - \beta q)}} \|f\|_{H^m} \right),$$

holds true for every $0 < \beta < \frac{(q-1)n}{q}$ and for any $f \in H^p \cap H^m$.

Commutator with BMO

Theorem ([BGS22])

Let Ω *satisfy conditions* (1) *and* (2). *If* $1 < r < p < \infty$ *and* $0 < l < n$ *are such that* $\frac{1}{p} = \frac{1}{r} - \frac{l}{n}$, *then there exists a constant* $C > 0$ *such that*

$$\|[b, T_\beta]f\|_{L^p} \leq C\|b\|_{BMO}\left(\|f\|_{L^p} + \frac{\beta^{n(1-\frac{1}{p})}}{\sqrt[p]{n(p-1)-\beta p}}\|f\|_{L^1} + \frac{\beta^l}{(l-\beta)^{1-\frac{l}{n}}}\|f\|_{L^r}\right)$$

holds true for all $b \in BMO$, $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, *and* $0 < \beta < l$.

Commutator with Lipschitz space

Theorem ([BGS22])

Let Ω *satisfy conditions* (1) *and* (2)*. If* $1 < p < q < \infty$ *and* $0 < \gamma < 1$ *be such that* $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$*, then there exists a constant* $C > 0$ *such that*

$$\|[b, T_\beta]f\|_{L^q} \leq C\|b\|_{Lip_\gamma}\left(\|f\|_{L^p} + \frac{\beta^{n(1-\frac{1}{q})-\gamma}}{\sqrt[q]{n(q-1)-(\beta-\gamma)q}}\|f\|_{L^1}\right)$$

holds true for all $b \in Lip_\gamma(\mathbb{R}^n)$*,* $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$*, and* $0 < \beta < n(1 - \frac{1}{q}) - \gamma$*.*

Commutator in Hardy space

Theorem ([BGS22])

Let Ω *satisfy conditions* (1) *and* (3) *for some* $0 < \alpha < 1$, *and let* $b \in Lip_\alpha(\mathbb{R}^n)$. *If* $\frac{n}{n+\alpha} < p \leq 1 < q < \infty$ *be such that* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *then there exists a constant* $C > 0$ *such that*

$$\|[b, T_\beta]f\|_{L^q} \leq C\|b\|_{Lip_\alpha}\left(\|f\|_{H^p} + \frac{\beta^{n(1-\frac{1}{q})-\alpha}}{\sqrt[q]{n(q-1)-(\beta-\alpha)q}}\|f\|_{L^1}\right)$$

holds true for all $f \in H^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, *and* $0 < \beta < n(1-\frac{1}{q}) - \alpha$.

Further open problem

1. Calderón's conjucture: Consider the bilinear Hilbert transform defined by

$$Hfg(x) = p.v. \int_{\mathbb{R}} f(x-y)g(x+y)\frac{dy}{y}$$

   Is H maps $L^p \times L^q$ into L^r for $1 < p, q \leq \infty, 1/p + 1/q = 1/r$?
2. This result is known to be true for $1 < p, q \leq \infty, 2/3 < r < \infty$.
3. Similarly one can consider *directional m-linear Hilbert transform* defined by

$$\mathcal{H}_\theta(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}} f_1(x - t\theta_1) \ldots f_m(x - t\theta_m)\frac{dt}{t}.$$

   Is $\mathcal{H}_\theta$ bounded from $L^{p_1}(\mathbb{R}^n) \times \ldots L^{p_m}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ uniformly in θ when $1 < p_1, \ldots, p_m, p < \infty$ satisfy $1/p_1 + \ldots 1/p_m = 1/p$?
4. The question has been answered so far only when $m = 2$ and $n = 1$.

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Thank You!