Biorthogonal family construction and its application to controllability

Graduate Student Seminar
Autumn 2022

Contents

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$$\begin{cases} U'(t) = AU(t) + BQ(t) \\ U(0) = U^0 \end{cases} \tag{1}$$

where, $A \in M_n(\mathbb{R}), B \in M_{n,m}(\mathbb{R}), U(t) \in \mathbb{R}^n, Q(t) \in \mathbb{R}^m$.

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(1) is null controllable in time T, if for any $U^0 \in \mathbb{R}^n$, $\exists$ control $Q(t) \in \mathbb{R}^m$ such that the corresponding solution satisfies $U(T) = 0$.

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e.g. $A = I_2, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : n = 2, m = 1$

The solution of this system is given by

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} u_1^0 e^t + \int_0^t e^{(t-s)} Q(s)\, ds \\ u_2^0 e^t \end{pmatrix}$$

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What if $B = I_2$ and $Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$? : $[m = n = 2]$

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Finally, one can define the control as $q(t) = u''(t) + u(t)$.

Introducing the systems

The linear system coupling KS-KdV equation with transport equation is given as:

$$\begin{cases} u_t + u_{xxxx} + u_{xxx} + u_{xx} + u = v_x, & (t,x) \in (0,T) \times (0,2\pi), \\ v_t + v_x + v = u_x, & (t,x) \in (0,T) \times (0,2\pi), \\ u(t,0) = u(t,2\pi), & t \in (0,T), \\ u^{(i)}(t,0) = u^{(i)}(t,2\pi), & t \in (0,T), i \in \{1,2,3\}, \\ v(t,0) = v(t,2\pi) + q(t), & t \in (0,T), \\ u(0,x) = u_0(x), v(0,x) = v_0(x), & x \in (0,2\pi), \end{cases} \tag{3}$$

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The coupled PDEs can be written as:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v_x - u_{xxxx} - u_{xxx} - u_{xx} - u \\ u_x - v_x - v \end{pmatrix}. \tag{4}$$

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Question:
We try to find a space X and some $T_0 \geq 0$ such that if we choose $(u_0, v_0) \in X$, and any $T > T_0$, we get existence of a control function $q_T \in L^2(0,T)$ such that the solution (u,v) satisfies $(u(T,x), v(T,x)) = (0,0)$.

Motivation for biorthogonal

- $L^2(0, 2\pi) = \{f : (0, 2\pi) \to \mathbb{C} : \int_0^{2\pi} |f|^2 < \infty\}$.

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- Asymptotic expression of eigenvalues:

$$\begin{cases} \lambda_0^{\pm} = -1 \\ \lambda_1^{+} = \lambda_{-1}^{+} = -1 + \sqrt{2}i \\ \lambda_1^{-} = \lambda_{-1}^{-} = -1 - \sqrt{2}i \\ \lambda_k^{+} = ik - 1 + O(k^{-1}), \text{ as } |k| \to \infty \\ \lambda_k^{-} = -k^4 - ik^3 + k^2 - 1 + O(k^{-1}), \text{ as } |k| \to \infty \end{cases}$$

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- This method relies on: **(Null controllability)** $\Leftrightarrow$ **(Solving a moment problem)**

$$\begin{cases} a_k^{+} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-\overline{\lambda_k^{+}} t} q\left(t + \frac{T}{2}\right) dt = b_k^{+}, \\ a_k^{-} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-\overline{\lambda_k^{-}} t} q\left(t + \frac{T}{2}\right) dt = b_k^{-}, \end{cases} \quad k \in \mathbb{Z} \setminus \{0, \pm 1\} : \text{Moment problem}$$

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- Need a family of functions **biorthogonal** to $\{e^{\lambda_k^{\pm} t}\}$ with proper L^2-estimates.

Biorthogonal family

Theorem (Existence of biorthogonal)

Let $T > 2\pi$. Then there exists a family $\{\Theta_k^{\pm}\}_{k\in\mathbb{Z}\setminus\{-1,0\}} \cup \{\Theta_0\}$ of functions in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ satisfying

$$\begin{cases} \int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_k^{\pm}(t) e^{-\overline{\lambda_l^{\pm}} t} dt = \delta_{kl}\delta_{\pm}, & l \in \mathbb{Z}\setminus\{-1\}, k \in \mathbb{Z}\setminus\{-1,0\}, \\ \int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_0(t) e^{-\overline{\lambda_l^{\pm}} t} dt = \delta_{0l}, & l \in \mathbb{Z}\setminus\{-1\}, \end{cases}$$

where,

$$\delta_{kl} = \begin{cases} 1, & \text{if } k = l \\ 0, & \text{if } k \neq l \end{cases} \text{ and } \delta_{\pm} = \begin{cases} 1, & \text{if sign on } \Theta_k \text{ and } \lambda_k \text{ is same} \\ 0, & \text{otherwise .} \end{cases}$$

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Moreover, we have following estimates on the obtained family

$$\begin{aligned} ||\Theta_k^{+}||_{L^2(\mathbb{R})} &\leq C|k|^4, & k \in \mathbb{Z}\setminus\{-1,0\}, \\ ||\Theta_k^{-}||_{L^2(\mathbb{R})} &\leq Ce^{-\frac{T}{2}(k^4-k^2)}, & k \in \mathbb{Z}\setminus\{-1,0\}, \\ ||\Theta_0||_{L^2(\mathbb{R})} &\leq C, \end{aligned}$$

where, C is a positive constant independent of k.

Preliminaries

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An entire function f is said to be of **exponential type at most** A if for any $\epsilon > 0$, there exist $B_\epsilon > 0$ such that

$$|f(z)| \leq B_\epsilon \, e^{(A+\epsilon)|z|}, \; z \in \mathbb{C}.$$

Contd.

Definition

An entire function f of exponential type π is said to be of sine type if

(i) the zeros of $f(z)$, say μ_k satisfies gap condition, i.e., there exist $\delta > 0$ such that $|\mu_k - \mu_l| > \delta$ for $k \neq l$, and

(ii) there exist positive constants C_1, C_2 and C_3 such that

$$C_1 e^{\pi|y|} \leq |f(x+iy)| \leq C_2 e^{\pi|y|}, \forall x, y \in \mathbb{R} \text{ with } |y| \geq C_3.$$

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Theorem (Properties of sine type function)

Let f be a sine type function, and let $\{\mu_k\}_{k\in\mathcal{I}}$ with $\mathcal{I} \subset \mathbb{Z}$ be its sequence of zeros. Then, we have:

(a) *for any $\epsilon > 0$, there exist constants $K_\epsilon, \tilde{K}_\epsilon > 0$ such that*

$$K_\epsilon e^{\pi|y|} \leq |f(x+iy)| \leq \tilde{K}_\epsilon e^{\pi|y|}, \ \textit{if } dist(x+iy, \{\mu_k\}) > \epsilon,$$

(b) *there exist some constants $K_1, K_2 > 0$ such that*

$$K_1 < |f'(\mu_k)| < K_2, \quad \forall k \in \mathcal{I}.$$

Contd.

Theorem (Paley-Wiener)

Let f be an entire function of exponential type A and suppose

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty.$$

Then there exists a function $\phi \in L^2(-A, A)$ with the following

$$f(z) = \int_{-A}^{A} e^{izt}\phi(t)dt, \; z \in \mathbb{C}.$$

Sketch of the proof

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$$\int_{\mathbb{R}} \left|\Psi_k^{\pm}(x)\right|^2 dx < \infty \text{ and } \int_{\mathbb{R}} |\Psi_0(x)|^2 \, dx < \infty. \tag{6}$$

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2. $$\int_{\mathbb{R}} \left|\Psi_k^{\pm}(x)\right|^2 dx < \infty \text{ and } \int_{\mathbb{R}} |\Psi_0(x)|^2 dx < \infty. \tag{6}$$

3. The following relation hold

$$\begin{cases} \Psi_k^{\pm}(i\mu_l^{\pm}) = \delta_{kl}\delta_{\pm}, & k \in \mathbb{Z} \setminus \{-1,0\}, \, l \in \mathbb{Z} \setminus \{-1\}, \\ \Psi_0(i\mu_l^{\pm}) = \delta_{0l}, & l \in \mathbb{Z} \setminus \{-1\}, \end{cases} \tag{7}$$

where, $\mu_0^+ = \mu_0^- = \mu_0 = -1$, $\mu_k^{\pm} = \overline{\lambda_k^{\pm}}$, $\forall k \in \mathbb{Z} \setminus \{0\}$.0

Contd.

Let us first introduce the following entire function which has simple zeros exactly at $i\mu_k^{\pm} = i\overline{\lambda_k^{\pm}}$:

$$P(z) = \left(1 - \frac{z}{i\mu_0}\right) \prod_{k\in\mathbb{Z}\setminus\{-1,0\}} \left(1 - \frac{z}{i\mu_k^{+}}\right) \prod_{k\in\mathbb{Z}\setminus\{-1,0\}} \left(1 - \frac{z}{i\mu_k^{-}}\right). \quad (8)$$

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Lemma

Let $\Lambda_k = k + d_k$, where $d_k = d + O(k^{-1})$, for $k \in \mathbb{Z} \setminus \{0\}$ as $|k| \to \infty$ for some constant $d \in \mathbb{C}$, and that $\Lambda_k \neq \Lambda_l$ for $k \neq l$. Then $f(z) = \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{\Lambda_k}\right)$ is an entire function of type sine.

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Define

$$P_1(z) = \prod_{k\in\mathbb{Z}\setminus\{-1\}} \left(1 - \frac{z}{i\mu_k^+}\right), \quad P_2(z) = \prod_{k\in\mathbb{Z}\setminus\{-1\}} \left(1 - \frac{z}{i\mu_k^-}\right),$$

Contd.

Let us first introduce the following entire function which has simple zeros exactly at $i\mu_k^{\pm} = i\overline{\lambda_k^{\pm}}$:

$$P(z) = \left(1 - \frac{z}{i\mu_0}\right) \prod_{k\in\mathbb{Z}\setminus\{-1,0\}} \left(1 - \frac{z}{i\mu_k^+}\right) \prod_{k\in\mathbb{Z}\setminus\{-1,0\}} \left(1 - \frac{z}{i\mu_k^-}\right). \tag{8}$$

Lemma

Let $\Lambda_k = k + d_k$, where $d_k = d + O(k^{-1})$, for $k \in \mathbb{Z} \setminus \{0\}$ as $|k| \to \infty$ for some constant $d \in \mathbb{C}$, and that $\Lambda_k \neq \Lambda_l$ for $k \neq l$. Then $f(z) = \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{\Lambda_k}\right)$ is an entire function of type sine.

Define

$$P_1(z) = \prod_{k\in\mathbb{Z}\setminus\{-1\}} \left(1 - \frac{z}{i\mu_k^+}\right), \quad P_2(z) = \prod_{k\in\mathbb{Z}\setminus\{-1\}} \left(1 - \frac{z}{i\mu_k^-}\right),$$

$$P(z) = \frac{P_1(z)P_2(z)}{\left(1 + \frac{z}{i}\right)}.$$

Contd.

For $k \in \mathbb{Z} \setminus \{-1, 0\}$, we define

$$\tilde{\delta}_k = \operatorname{sgn}(k) \sqrt[4]{-\mu_k^-} = k + b_k,$$

where $b_k = -\frac{i}{4} + O(k^{-1})$ as $|k| \to \infty$ and let $\tilde{\delta}_0 = 1$.

Contd.

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$$Q_1(z) = \prod_{k \in \mathbb{Z} \setminus \{-1\}} \left(1 - \frac{z}{\tilde{\delta}_k}\right),$$

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$$Q_2(z) = \prod_{k \in \mathbb{Z} \setminus \{-1\}} \left(1 + \frac{z^4}{\mu_k^-}\right).$$

Thus, we get the following relations:

$$Q_2(z) = Q_1(z)Q_1(-z)Q_1(iz)Q_1(-iz), \tag{9}$$

$$P_2(z) = Q_2(e^{i\frac{\pi}{8}}\sqrt[4]{z}). \tag{10}$$

Contd.

Lemma (Estimates on P)

Let P be the canonical product defined in (8). Then P is an entire function of exponential type at most π, which satisfies the following estimates for some positive constants C, C_1, C_2 independent of k:

$$|P(x)| \leq C\,|x+i|^{-1}\, e^{2\pi\left(\cos(\frac{\pi}{8})+\sin(\frac{\pi}{8})\right)|x|^{\frac{1}{4}}}, \quad \forall x \in \mathbb{R}, \tag{11}$$

$$\left|P'(i\mu_k^+)\right| \geq C_1|k|^{-1}e^{2\pi\left(\cos(\frac{\pi}{8})+\sin(\frac{\pi}{8})\right)|k|^{\frac{1}{4}}}, \quad k \in \mathbb{Z}\setminus\{-1,0\} \tag{12}$$

$$\left|P'(i\mu_k^-)\right| \geq C_2|k|^{-7}e^{\pi\left(k^4-k^2+2|k|\right)}, \quad k \in \mathbb{Z}\setminus\{-1,0\}. \tag{13}$$

Contd.

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Lemma (Estimate for the multiplier M)

There exist an entire function, M of exponential type at most $a\pi$, where $a = \left(\frac{T}{2\pi}-1\right) > 0$ which satisfies the following bound for some $C, \widetilde{C}, \widehat{C} > 0$:

$$|M(x)| \leq C\,|x| \exp\left[-2\pi\left(\cos\left(\frac{\pi}{8}\right)+\sin\left(\frac{\pi}{8}\right)\right)|x|^{\frac{1}{4}}\right], \quad \forall x \in \mathbb{R}, \tag{14}$$

$$|M(i\mu_k^+)| \geq \widetilde{C}|k|^{-3}\exp\left[-2\pi\left(\cos\left(\frac{\pi}{8}\right)+\sin\left(\frac{\pi}{8}\right)\right)|k|^{\frac{1}{4}}\right], \quad \forall k \in \mathbb{Z}\setminus\{-1,0\}, \tag{15}$$

$$|M(i\mu_k^-)| \geq \widehat{C}\,|k|^{-8}\exp\left[\pi a(k^4-k^2)-c|k|\right], \quad \forall k \in \mathbb{Z}\setminus\{-1,0\}. \tag{16}$$

Contd.

Now, we can finally define $\Psi_k^{\pm}$ as

$$\Psi_k^{\pm}(z) = \frac{P(z)}{P'(i\mu_k^{\pm})(z - i\mu_k^{\pm})} \frac{M(z)}{M(i\mu_k^{\pm})}, \ k \in \mathbb{Z} \setminus \{-1, 0\}$$

and

$$\Psi_0(z) = \frac{P(z)}{P'(-i)(z+i)} \frac{M(z)}{M(-i)}.$$

Clearly, Ψ_k is an entire function of exponential type at most $\pi + a\pi = \frac{T}{2}$ and satisfies

$$\Psi_k^{\pm}(i\mu_l^{\pm}) = \delta_{kl}\,\delta_{\pm}, \quad \forall\, l \in \mathbb{Z} \setminus \{-1\} \text{ and } k \in \mathbb{Z} \setminus \{-1, 0\},$$
$$\Psi_0(i\mu_l^{\pm}) = \delta_{0l}, \quad \forall\, l \in \mathbb{Z} \setminus \{-1\}.$$

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It can be seen that

$$|P(x)M(x)| \leq C.$$

Contd.

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For $k \in \mathbb{Z} \setminus \{-1, 0\}$, $\Psi_k^{+}, \Psi_k^{-} \in L^2(\mathbb{R})$ with

$$||\Psi_k^{+}||_{L^2(\mathbb{R})} \leq C\,|k|^4,$$
$$||\Psi_k^{-}||_{L^2(\mathbb{R})} \leq C|k|^{15} e^{-\frac{T}{2}(k^4 - k^2)}.$$

Contd.

Now, we can finally define $\Psi_k^{\pm}$ as

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$$||\Psi_k^-||_{L^2(\mathbb{R})} \leq C|k|^{15} e^{-\frac{T}{2}(k^4 - k^2)}.$$

Also,

$$|\Psi_0(x)| \leq \frac{C}{|x+i|},$$

and so $\Psi_0 \in L^2(\mathbb{R})$ as well.

Controllability Results

$$
\begin{cases}
u_t + u_{xxxx} + u_{xxx} + u_{xx} = v_x + h_1, & (t,x) \in (0,T) \times (0,2\pi), \\
v_t + v_x = u_x + h_2, & (t,x) \in (0,T) \times (0,2\pi), \\
u(t,0) = u(t,2\pi) + q_1(t), & t \in (0,T), \\
u^{(i)}(t,0) = u^{(i)}(t,2\pi), & t \in (0,T), i \in \{1,2,3\}, \\
v(t,0) = v(t,2\pi) + q_2(t), & t \in (0,T), \\
u(0,x) = u_0(x), v(0,x) = v_0(x), & x \in (0,2\pi),
\end{cases}
\tag{17}
$$

Controllability Results

$$\begin{cases} u_t + u_{xxxx} + u_{xxx} + u_{xx} = v_x + h_1, & (t,x) \in (0,T) \times (0,2\pi), \\ v_t + v_x = u_x + h_2, & (t,x) \in (0,T) \times (0,2\pi), \\ u(t,0) = u(t,2\pi) + q_1(t), & t \in (0,T), \\ u^{(i)}(t,0) = u^{(i)}(t,2\pi), & t \in (0,T), i \in \{1,2,3\}, \\ v(t,0) = v(t,2\pi) + q_2(t), & t \in (0,T), \\ u(0,x) = u_0(x), v(0,x) = v_0(x), & x \in (0,2\pi), \end{cases} \tag{17}$$

Theorem (Null controllability)

Let $(u_0, v_0) \in X$. Then for any time $T > T_0 = 2\pi$, there exists a control function $F \in Z$ such that the solution of (17) satisfies $(u(T,x), v(T,x)) = (0,0)$, where X, F and Z depend on the control system in consideration and are given explicitly in the table below:

S. No.	Space of initial data, X	Control function, F
(a)	$\{(u_0, v_0) \in \mathcal{H}_p^s(0,2\pi) : \langle v_0, 1\rangle_{L^2} = 0\}$, $s > 6.5$	$h_1(t,x) = \mathbb{1}_\omega f(x)g(t) \in L^2(0,T; L^2(\omega))$
(b)	$\{(u_0, v_0) \in \mathcal{H}_p^s(0,2\pi) : \langle u_0, 1\rangle_{L^2} = 0\}$, $s > 3.5$	$h_2(t,x) = \mathbb{1}_\omega f(x)g(t) \in L^2(0,T; L^2(\omega))$
(c)	$\{(u_0, v_0) \in \mathcal{H}_p^s(0,2\pi) : \langle u_0, 1\rangle_{L^2} = 0\}$, $s > 3.5$	$q_1 \in L^2(0,T)$
(d)	$\{(u_0, v_0) \in \mathcal{H}_p^s(0,2\pi) : \langle u_0, 1\rangle_{L^2} = \langle v_0, 1\rangle_{L^2} = 0\}$, $s > 1.5$	$q_2 \in L^2(0,T)$

where, $\mathcal{H}_p^s = \left\{(u,\eta) \in H_p^s(0,2\pi) \times H_p^{s+3}(0,2\pi) : \left\langle u, e^{\pm ix}\right\rangle_{L^2(0,2\pi)} = \left\langle \eta, e^{\pm ix}\right\rangle_{L^2(0,2\pi)} = 0\right\}$.

References

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Thank You!