THE LAPLACIAN SPECTRA OF COMMUTING GRAPHS

An approach to derive the complete list of eigenpairs of the commuting graphs from the group properties

Samiron Parui
(Joint work with Gargi Ghosh)

IISER Kolkata

April 30, 2022

IISER KOLKATA

Outline

1. Fundamentals
   - Graphs and Group
   - Commuting Graph
   - Center and Centralizer
   - Matrices Associated to Graphs

2. The Laplacian spectra of Commuting Graphs
   - The eigenvalue 0
   - Abelian group
   - Non-abelian group

Graph and Group

Group $(G, .)$

- $G(\neq \emptyset)$
- $G \times G \to G$
  $(g_1, g_2) \mapsto g_1 \cdot g_2 \in G$
  - Associative $(g_1.g_2).g_3 = g_1.(g_2.g_3)$
  - Identity $g.e = e.g = g, \forall g$
  - Inverse $g.f = f.g = e, \forall g$

Graph $\Gamma(V, E)$

- Set of vertices, V
  - $V \neq \emptyset$
  - $v \in V \implies v$ is a vertex
- Set of edges, E
  - $e \in E \implies e = \{u, v\}$ for some $u, v \in V$
  - Each element is called an edge.

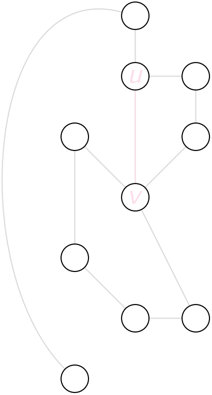
Graph and Group

Group $(G, .)$

- $G(\neq \emptyset)$
- $G \times G \to G$
  $(g_1, g_2) \mapsto g_1 \cdot g_2 \in G$
  - Associative $(g_1.g_2).g_3 = g_1.(g_2.g_3)$
  - Identity $g.e = e.g = g, \forall g$
  - Inverse $g.f = f.g = e, \forall g$

Graph $\Gamma(V, E)$

- Set of vertices, V
  - $V \neq \emptyset$
  - $v \in V \implies v$ is a vertex
- Set of edges, E
  - $e \in E \implies e = \{u, v\}$ for some $u, v \in V$
  - Each element is called an edge.



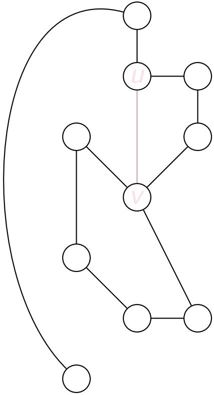
Graph and Group

Group $(G, .)$

- $G(\neq \emptyset)$
- $G \times G \to G$
  $(g_1, g_2) \mapsto g_1 \cdot g_2 \in G$
  - Associative $(g_1.g_2).g_3 = g_1.(g_2.g_3)$
  - Identity $g.e = e.g = g, \forall g$
  - Inverse $g.f = f.g = e, \forall g$

Graph $\Gamma(V, E)$

- Set of vertices, V
  - $V \neq \emptyset$
  - $v \in V \implies v$ is a vertex
- Set of edges, E
  - $e \in E \implies e = \{u, v\}$ for some $u, v \in V$
  - Each element is called an edge.



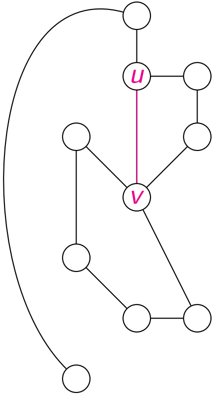
Graph and Group

Group $(G, .)$

- $G(\neq \emptyset)$
- $G \times G \to G$
  $(g_1, g_2) \mapsto g_1 \cdot g_2 \in G$
  - Associative $(g_1.g_2).g_3 = g_1.(g_2.g_3)$
  - Identity $g.e = e.g = g, \forall g$
  - Inverse $g.f = f.g = e, \forall g$

Graph $\Gamma(V, E)$

- Set of vertices, V
  - $V \neq \emptyset$
  - $v \in V \implies v$ is a vertex
- Set of edges, E
  - $e \in E \implies e = \{u, v\}$ for some $u, v \in V$
  - Each element is called an edge.



Commuting Graph

- A finite group G.
- A graph $\Gamma = (V, E)$ is such that
  - $V = G$,
  - $E = \{(u, v) : u, v \in G \text{ and } uv = vu, v \neq u\}$.
- The graph Γ is called the Commuting Graph of the group G. Commuting graph is denoted by $\mathcal{C}_G$.

Example

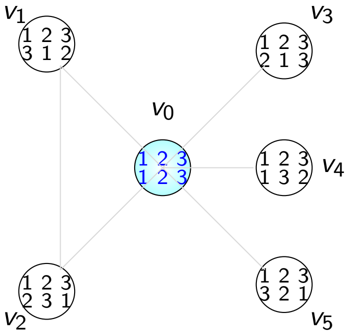


Figure: $\mathcal{C}_{\mathfrak{S}_3}$: Commuting Graph associated to $\mathfrak{S}_3$

Commuting Graph

- A finite group G.
- A graph $\Gamma = (V, E)$ is such that
  - $V = G$,
  - $E = \{(u, v) : u, v \in G \text{ and } uv = vu, v \neq u\}$.
- The graph Γ is called the Commuting Graph of the group G. Commuting graph is denoted by $\mathcal{C}_G$.

Example

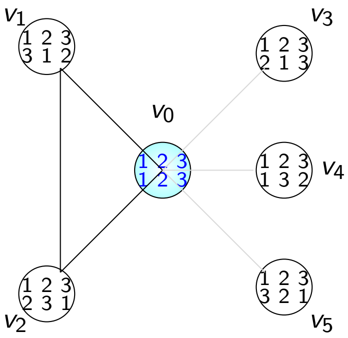


Figure: $\mathcal{C}_{\mathfrak{S}_3}$: Commuting Graph associated to $\mathfrak{S}_3$

Commuting Graph

- A finite group G.
- A graph $\Gamma = (V, E)$ is such that
  - $V = G$,
  - $E = \{(u, v) : u, v \in G \text{ and } uv = vu, v \neq u\}$.
- The graph Γ is called the Commuting Graph of the group G. Commuting graph is denoted by $\mathcal{C}_G$.

Example

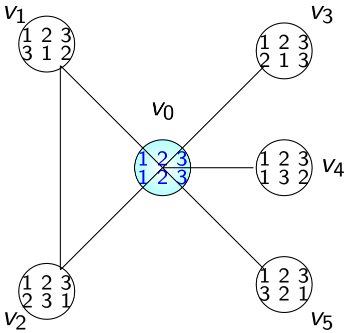


Figure: $\mathcal{C}_{\mathfrak{S}_3}$: Commuting Graph associated to $\mathfrak{S}_3$

Center and Centralizer

Center of the group G

$Z(G) = \{u \in G : ua = au \forall a \in G\}$.

Centralizer of an element

$C(v) = \{u \in G : uv = vu\}$.

Figure: Center and Centralizer

Example

$Z(\mathfrak{S}_3) = \{v_0\} = \{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right)\}$

Example

In $\mathfrak{S}_3$,

- $C(v_1) = C(v_2) = \{v_0, v_1, v_2\}$ and
- $C(v_3) = \{v_0, v_3\}$, $C(v_4) = \{v_0, v_4\}$, $C(v_5) = \{v_0, v_5\}$.

Center and Centralizer

Center of the group G

$Z(G) = \{u \in G : ua = au \forall a \in G\}$.

Centralizer of an element

$C(v) = \{u \in G : uv = vu\}$.

Figure: Center and Centralizer

Example

$Z(\mathfrak{S}_3) = \{v_0\} = \{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right)\}$

Example

In $\mathfrak{S}_3$,

- $C(v_1) = C(v_2) = \{v_0, v_1, v_2\}$ and
- $C(v_3) = \{v_0, v_3\}$, $C(v_4) = \{v_0, v_4\}$, $C(v_5) = \{v_0, v_5\}$.

Center and Centralizer

Center of the group G

$Z(G) = \{u \in G : ua = au \forall a \in G\}$.

Centralizer of an element

$C(v) = \{u \in G : uv = vu\}$.

Figure: Center and Centralizer

Example

$Z(\mathfrak{S}_3) = \{v_0\} = \{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right)\}$

Example

In $\mathfrak{S}_3$,

- $C(v_1) = C(v_2) = \{v_0, v_1, v_2\}$ and
- $C(v_3) = \{v_0, v_3\}$, $C(v_4) = \{v_0, v_4\}$, $C(v_5) = \{v_0, v_5\}$.

Center and Centralizer

Center of the group G

$Z(G) = \{u \in G : ua = au \forall a \in G\}$.

Centralizer of an element

$C(v) = \{u \in G : uv = vu\}$.

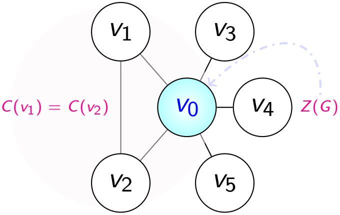


Figure: Center and Centralizer

Example

$Z(\mathfrak{S}_3) = \{v_0\} = \{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}\}$

Example

In $\mathfrak{S}_3$,

- $C(v_1) = C(v_2) = \{v_0, v_1, v_2\}$ and
- $C(v_3) = \{v_0, v_3\}$, $C(v_4) = \{v_0, v_4\}$, $C(v_5) = \{v_0, v_5\}$.

Center and Centralizer

Center of the group G

$Z(G) = \{u \in G : ua = au \forall a \in G\}$.

Centralizer of an element

$C(v) = \{u \in G : uv = vu\}$.

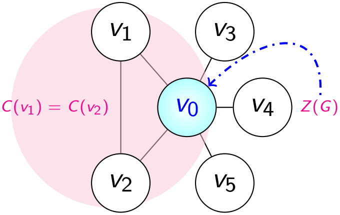


Figure: Center and Centralizer

Example

$Z(\mathfrak{S}_3) = \{v_0\} = \{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}\}$

Example

In $\mathfrak{S}_3$,

- $C(v_1) = C(v_2) = \{v_0, v_1, v_2\}$ and
- $C(v_3) = \{v_0, v_3\}, \ C(v_4) = \{v_0, v_4\}, \ C(v_5) = \{v_0, v_5\}$.

Matrices

- $\Gamma(V,E)$: Any undirected graph
- $A = (a_{uv})_{u,v\in V}$: A square matrix of order $|V|$ defined by

$$a_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

- $L_\Gamma = D - A$: Laplacian matrix
- $\mathbb{R}^V$: The set of all functions from V to $\mathbb{R}$

$$(L_\Gamma x)(u) = \sum_{v\in V} a_{uv}(x(u)-x(v)) \text{ for } x \in \mathbb{R}^V \text{ and } u \in V.$$

- If $\Gamma(V,E) = \mathcal{C}_G$: The Laplacian operator is given by

$$(L_\Gamma x)(u) = \sum_{v\in C(u)} (x(u)-x(v)), \text{ for } x \in \mathbb{R}^G \text{ and } u \in G.$$

Figure: Graph G

$L =$

	0	1	2	3	4	5
0	5	-1	-1	-1	-1	-1
1	-1	2	-1	0	0	0
2	-1	-1	2	0	0	0
3	-1	0	0	1	0	0
4	-1	0	0	0	1	0
5	-1	0	0	0	0	1

Matrices

- $\Gamma(V, E)$: Any undirected graph
- $A = (a_{uv})_{u,v \in V}$: A square matrix of order $|V|$ defined by

$$a_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

- $L_\Gamma = D - A$: Laplacian matrix
- $\mathbb{R}^V$: The set of all functions from V to $\mathbb{R}$

$$(L_\Gamma x)(u) = \sum_{v \in V} a_{uv}(x(u) - x(v)) \text{ for } x \in \mathbb{R}^V \text{ and } u \in V.$$

- If $\Gamma(V, E) = \mathcal{C}_G$: The Laplacian operator is given by

$$(L_\Gamma x)(u) = \sum_{v \in C(u)} (x(u) - x(v)), \text{ for } x \in \mathbb{R}^G \text{ and } u \in G.$$

Figure: Graph G

$L =$

	0	1	2	3	4	5
0	5	-1	-1	-1	-1	-1
1	-1	2	-1	0	0	0
2	-1	-1	2	0	0	0
3	-1	0	0	1	0	0
4	-1	0	0	0	1	0
5	-1	0	0	0	0	1

Matrices

- $\Gamma(V, E)$: Any undirected graph
- $A = (a_{uv})_{u,v \in V}$: A square matrix of order $|V|$ defined by

$$a_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

- $L_\Gamma = D - A$: Laplacian matrix
- $\mathbb{R}^V$: The set of all functions from V to $\mathbb{R}$

$$(L_\Gamma x)(u) = \sum_{v \in V} a_{uv}(x(u) - x(v)) \text{ for } x \in \mathbb{R}^V \text{ and } u \in V.$$

- If $\Gamma(V, E) = \mathcal{C}_G$: The Laplacian operator is given by

$$(L_\Gamma x)(u) = \sum_{v \in C(u)} (x(u) - x(v)), \text{ for } x \in \mathbb{R}^G \text{ and } u \in G.$$

Figure: Graph G

$L =$

	0	1	2	3	4	5
0	5	-1	-1	-1	-1	-1
1	-1	2	-1	0	0	0
2	-1	-1	2	0	0	0
3	-1	0	0	1	0	0
4	-1	0	0	0	1	0
5	-1	0	0	0	0	1

Matrices

- $\Gamma(V,E)$: Any undirected graph
- $A = (a_{uv})_{u,v\in V}$: A square matrix of order $|V|$ defined by

$$a_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

- $L_\Gamma = D - A$: Laplacian matrix
- $\mathbb{R}^V$: The set of all functions from V to $\mathbb{R}$

$$(L_\Gamma x)(u) = \sum_{v\in V} a_{uv}(x(u) - x(v)) \text{ for } x \in \mathbb{R}^V \text{ and } u \in V.$$

- If $\Gamma(V,E) = \mathcal{C}_G$: The Laplacian operator is given by

$$(L_\Gamma x)(u) = \sum_{v\in C(u)} (x(u) - x(v)), \text{ for } x \in \mathbb{R}^G \text{ and } u \in G.$$

Figure: Graph G

$L =$

	0	1	2	3	4	5
0	5	-1	-1	-1	-1	-1
1	-1	2	-1	0	0	0
2	-1	-1	2	0	0	0
3	-1	0	0	1	0	0
4	-1	0	0	0	1	0
5	-1	0	0	0	0	1

Matrices

- $\Gamma(V, E)$: Any undirected graph
- $A = (a_{uv})_{u,v \in V}$: A square matrix of order $|V|$ defined by

$$a_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

- $L_\Gamma = D - A$: Laplacian matrix
- $\mathbb{R}^V$: The set of all functions from V to $\mathbb{R}$

$$(L_\Gamma x)(u) = \sum_{v \in V} a_{uv}(x(u) - x(v)) \text{ for } x \in \mathbb{R}^V \text{ and } u \in V.$$

- If $\Gamma(V, E) = \mathcal{C}_G$: The Laplacian operator is given by

$$(L_\Gamma x)(u) = \sum_{v \in C(u)} (x(u) - x(v)), \text{ for } x \in \mathbb{R}^G \text{ and } u \in G.$$

Figure: Graph G

$L =$

	0	1	2	3	4	5
0	5	-1	-1	-1	-1	-1
1	-1	2	-1	0	0	0
2	-1	-1	2	0	0	0
3	-1	0	0	1	0	0
4	-1	0	0	0	1	0
5	-1	0	0	0	0	1

Matrices

- $\Gamma(V, E)$: Any undirected graph
- $A = (a_{uv})_{u,v \in V}$: A square matrix of order $|V|$ defined by

$$a_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

- $L_\Gamma = D - A$: Laplacian matrix
- $\mathbb{R}^V$: The set of all functions from V to $\mathbb{R}$

$$(L_\Gamma x)(u) = \sum_{v \in V} a_{uv}(x(u) - x(v)) \text{ for } x \in \mathbb{R}^V \text{ and } u \in V.$$

- If $\Gamma(V, E) = \mathcal{C}_G$: The Laplacian operator is given by

$$(L_\Gamma x)(u) = \sum_{v \in C(u)} (x(u) - x(v)), \text{ for } x \in \mathbb{R}^G \text{ and } u \in G.$$

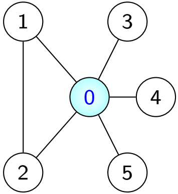


Figure: Graph G

$L =$

	0	1	2	3	4	5
0	5	-1	-1	-1	-1	-1
1	-1	2	-1	0	0	0
2	-1	-1	2	0	0	0
3	-1	0	0	1	0	0
4	-1	0	0	0	1	0
5	-1	0	0	0	0	1

Matrices

- $\Gamma(V,E)$: Any undirected graph
- $A = (a_{uv})_{u,v\in V}$: A square matrix of order $|V|$ defined by

$$a_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

- $L_\Gamma = D - A$: Laplacian matrix
- $\mathbb{R}^V$: The set of all functions from V to $\mathbb{R}$

$$(L_\Gamma x)(u) = \sum_{v\in V} a_{uv}(x(u) - x(v)) \text{ for } x \in \mathbb{R}^V \text{ and } u \in V.$$

- If $\Gamma(V,E) = \mathcal{C}_G$: The Laplacian operator is given by

$$(L_\Gamma x)(u) = \sum_{v\in C(u)} (x(u) - x(v)), \text{ for } x \in \mathbb{R}^G \text{ and } u \in G.$$

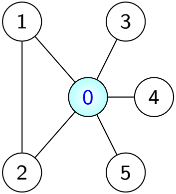


Figure: Graph G

$L =$

	0	1	2	3	4	5
0	5	-1	-1	-1	-1	-1
1	-1	2	-1	0	0	0
2	-1	-1	2	0	0	0
3	-1	0	0	1	0	0
4	-1	0	0	0	1	0
5	-1	0	0	0	0	1

The Constant Eigenvector

- $\chi_V : V \to \mathbb{R}$ defined by $\chi_V(v) = 1\ \forall v \in V$.
  - $\langle \chi_V \rangle \implies \{f : V \to \mathbb{R} | f = \text{constant}\}$.
  - $L_\Gamma \chi_V = 0$
  - If $\Gamma(V, E)$ is connected then $\langle \chi_V \rangle$ is the eigenspace of the eigenvalue 0.
- $\mathcal{C}_G$ is connected.

Therefore, 0 is an eigenvalue of L_Γ with the eigenspace $\langle \chi_V \rangle$.

The Constant Eigenvector

- $\chi_V : V \to \mathbb{R}$ defined by $\chi_V(v) = 1 \; \forall v \in V$.
  - $\langle \chi_V \rangle \Longrightarrow \{f : V \to \mathbb{R} | f = \text{constant}\}$.
  - $L_\Gamma \chi_V = 0$
  - If $\Gamma(V, E)$ is connected then $\langle \chi_V \rangle$ is the eigenspace of the eigenvalue 0.
- $\mathcal{C}_G$ is connected.

Therefore, 0 is an eigenvalue of L_Γ with the eigenspace $\langle \chi_V \rangle$.

The Constant Eigenvector

- $\chi_V : V \to \mathbb{R}$ defined by $\chi_V(v) = 1 \ \forall v \in V$.
  - $\langle \chi_V \rangle \implies \{f : V \to \mathbb{R} | f = \text{constant}\}$.
  - $L_\Gamma \chi_V = 0$
  - If $\Gamma(V, E)$ is connected then $\langle \chi_V \rangle$ is the eigenspace of the eigenvalue 0.
- $\mathcal{C}_G$ is connected.

Therefore, 0 is an eigenvalue of L_Γ with the eigenspace $\langle \chi_V \rangle$.

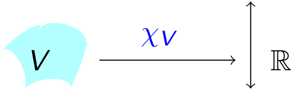
The Constant Eigenvector

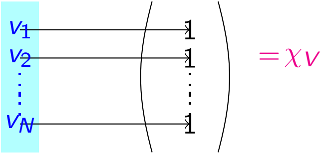
- $\chi_V : V \to \mathbb{R}$ defined by $\chi_V(v) = 1\ \forall v \in V$.
  - $\langle \chi_V \rangle \implies \{f : V \to \mathbb{R} | f = \text{constant}\}$.
  - $L_\Gamma \chi_V = 0$
  - If $\Gamma(V, E)$ is connected then $\langle \chi_V \rangle$ is the eigenspace of the eigenvalue 0.
- $\mathcal{C}_G$ is connected.

Therefore, 0 is an eigenvalue of L_Γ with the eigenspace $\langle \chi_V \rangle$.

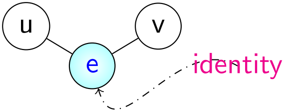
The Constant Eigenvector

- $\chi_V : V \to \mathbb{R}$ defined by $\chi_V(v) = 1 \ \forall v \in V$.
  - $\langle \chi_V \rangle \implies \{f : V \to \mathbb{R} | f = \text{constant}\}$.
  - $L_\Gamma \chi_V = 0$
  - If $\Gamma(V, E)$ is connected then $\langle \chi_V \rangle$ is the eigenspace of the eigenvalue 0.
- $\mathcal{C}_G$ is connected.





Therefore, 0 is an eigenvalue of L_Γ with the eigenspace $\langle \chi_V \rangle$.



When the G is Abelian

- G is abelian $\implies \mathcal{C}_G = K_{|G|}$
- The complete list of eigenvalue and eigenvector

Eigenvalue	Eigenvector	Eigenspace
0	χ_V	$\langle \chi_V \rangle$
$\|G\|$	$y_v(u) = \begin{cases} \|V\| - 1 & \text{if } u = v, \\ -1 & \text{otherwise.} \end{cases}$	$\langle \{y_v\}_{v \in V \setminus \{v_0\}} \rangle$

- For any $n \in \mathbb{N}$, $K_n = \mathcal{C}_{\mathbb{Z}_n}$.

When the G is Abelian

- G is abelian $\implies \mathcal{C}_G = K_{|G|}$
- The complete list of eigenvalue and eigenvector

Eigenvalue	Eigenvector	Eigenspace
0	χ_V	$\langle \chi_V \rangle$
$\|G\|$	$y_v(u) = \begin{cases} \|V\| - 1 & \text{if } u = v, \\ -1 & \text{otherwise.} \end{cases}$	$\langle \{y_v\}_{v \in V \setminus \{v_0\}} \rangle$

- For any $n \in \mathbb{N}$, $K_n = \mathcal{C}_{\mathbb{Z}_n}$.

When the G is Abelian

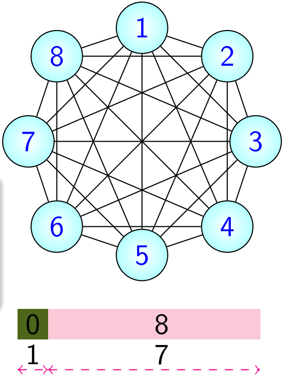
- G is abelian $\implies \mathcal{C}_G = K_{|G|}$
- The complete list of eigenvalue and eigenvector

Eigenvalue	Eigenvector	Eigenspace
0	χ_V	$\langle \chi_V \rangle$
$\lvert G \rvert$	$y_v(u) = \begin{cases} \lvert V \rvert - 1 & \text{if } u = v, \\ -1 & \text{otherwise.} \end{cases}$	$\langle \{y_v\}_{v \in V \setminus \{v_0\}} \rangle$

- For any $n \in \mathbb{N}$, $K_n = \mathcal{C}_{\mathbb{Z}_n}$.

When the G is Abelian

- G is abelian $\implies \mathcal{C}_G = K_{|G|}$
- The complete list of eigenvalue and eigenvector



Eigenvalue	Eigenvector	Eigenspace
0	χ_V	$\langle \chi_V \rangle$
$\lvert G\rvert$	$y_v(u) = \begin{cases} \lvert V\rvert - 1 & \text{if } u = v, \\ -1 & \text{otherwise.} \end{cases}$	$\langle \{y_v\}_{v \in V \setminus \{v_0\}} \rangle$

- For any $n \in \mathbb{N}$, $K_n = \mathcal{C}_{\mathbb{Z}_n}$.

Non-Abelian Group

Notation: $\sigma(L)$ = Set of all eigenvalues of the matrix L.

- **Question:** We have seen $0, |G| \in \sigma(L_{\mathcal{C}_G})$ for any commutative group G. Is it also true for a non-commutative group?
- **Answer:** Yes.

Lemma

$G \implies$ a non-abelian group

- $0 \in \sigma(L_{\mathcal{C}_G})$
  Multiplicity $= 1$
  Eigenvector $= \chi_V = (1, \ldots, 1)^t$
- $|G| \in \sigma(L_{\mathcal{C}_G})$
  Multiplicity at least $|Z(G)|$.

Figure: Unknown eigenvalues:grey

proof
Back to complete Laplacian spectra

Non-Abelian Group

Notation: $\sigma(L) =$ Set of all eigenvalues of the matrix L.

- **Question:** We have seen $0, |G| \in \sigma(L_{\mathcal{C}_G})$ for any commutative group G. Is it also true for a non-commutative group?
- **Answer:** Yes.

Lemma

$G \implies$ a non-abelian group

- $0 \in \sigma(L_{\mathcal{C}_G})$
  Multiplicity $= 1$
  Eigenvector $= \chi_V = (1, \ldots, 1)^t$
- $|G| \in \sigma(L_{\mathcal{C}_G})$
  Multiplicity at least $|Z(G)|$.

Figure: Unknown eigenvalues:grey

proof | Back to complete Laplacian spectra

Non-Abelian Group

Notation: $\sigma(L)$ = Set of all eigenvalues of the matrix L.

- **Question:** We have seen $0, |G| \in \sigma(L_{\mathcal{C}_G})$ for any commutative group G. Is it also true for a non-commutative group?
- **Answer:** Yes.

Lemma

$G \implies$ a non-abelian group

- $0 \in \sigma(L_{\mathcal{C}_G})$
  Multiplicity $= 1$
  Eigenvector $= \chi_V = (1, \ldots, 1)^{\mathrm{t}}$
- $|G| \in \sigma(L_{\mathcal{C}_G})$
  Multiplicity at least $|Z(G)|$.

Figure: Unknown eigenvalues:grey

proof Back to complete Laplacian spectra

Non-Abelian Group

Notation: $\sigma(L)$ = Set of all eigenvalues of the matrix L.

- **Question:** We have seen $0, |G| \in \sigma(L_{\mathcal{C}_G})$ for any commutative group G. Is it also true for a non-commutative group?
- **Answer:**Yes.

Lemma

$G \implies$ a non-abelian group

- $0 \in \sigma(L_{\mathcal{C}_G})$
  Multiplicity $= 1$
  Eigenvector $= \chi_V = (1, \ldots, 1)^t$
- $|G| \in \sigma(L_{\mathcal{C}_G})$
  Multiplicity at least $|Z(G)|$.

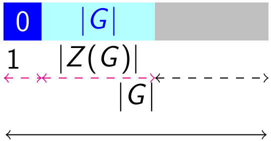


Figure: Unknown eigenvalues:grey

proof
Back to complete Laplacian spectra

Non-Abelian Group

The following result indicates another eigenvalue of L, provided a mild condition is imposed on an element of $G \setminus Z(G)$.

Theorem

$G \implies$ non-abelian finite group with $u \in G \setminus Z(G)$ such that for all $v \in G \setminus Z(G)$,

$$\text{either } C(u) = C(v) \quad \text{or } C(u) \cap C(v) = Z(G) \qquad (1)$$

then $|C(u)| \in \sigma(L_{\mathcal{C}_G})$
Multiplicity: at least $|C(u) \setminus Z(G)| - 1$.

Either or

Figure: Unknown eigenvalues:grey

Back to complete Laplacian spectra

Non-Abelian Group

The following result indicates another eigenvalue of L, provided a mild condition is imposed on an element of $G \setminus Z(G)$.

Theorem

$G \implies$ non-abelian finite group with $u \in G \setminus Z(G)$ such that for all $v \in G \setminus Z(G)$,

$$\text{either } C(u) = C(v) \\ \text{or } C(u) \cap C(v) = Z(G) \qquad (1)$$

then $|C(u)| \in \sigma(L_{\mathcal{C}_G})$
Multiplicity: at least $|C(u) \setminus Z(G)| - 1$.

Figure: Unknown eigenvalues:grey

Back to complete Laplacian spectra

Non-Abelian Group

The following result indicates another eigenvalue of L, provided a mild condition is imposed on an element of $G \setminus Z(G)$.

Theorem

$G \implies$ non-abelian finite group with $u \in G \setminus Z(G)$ such that for all $v \in G \setminus Z(G)$,

$$\text{either } C(u) = C(v)$$
$$\text{or } C(u) \cap C(v) = Z(G) \qquad (1)$$

then $|C(u)| \in \sigma(L_{\mathcal{C}_G})$
Multiplicity: at least $|C(u) \setminus Z(G)| - 1$.

Either or

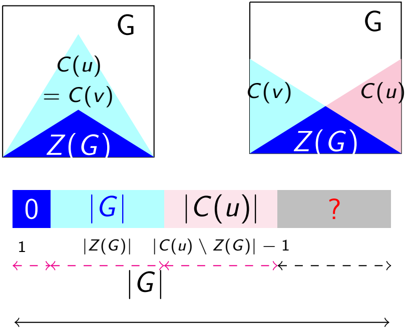


Figure: Unknown eigenvalues:grey

Back to complete Laplacian spectra

Condition: either $C(u) = C(v)$ or $C(u) \cap C(v) = Z(G)$

Suppose that $u \in G \setminus Z(G)$ satisfies the condition.

1. There exists at least one $v \in G$, other than u, such that $v \notin Z(G)$ and $C(u) \cap C(v) = Z(G)$.
2. $C(v) = C(u)$ for all $v \in C(u) \setminus Z(G)$.
3. If $C(u) = C(v)$ then we do not count the contribution of u and v separately. All the $v \in C(u) \setminus Z(G)$ togetherly contribute the eigenvalue $|C(u)|$ of multiplicity $|C(u) \setminus Z(G)|$.
4. For any $w \in G \setminus C(u)$, $C(u) \cap C(w) = Z(G)$. Suppose not, then $C(u) = C(w)$ but it contradicts to the assumption $w \notin C(u)$.

Condition: either $C(u) = C(v)$ or $C(u) \cap C(v) = Z(G)$

Suppose that $u \in G \setminus Z(G)$ satisfies the condition.

1. There exists at least one $v \in G$, other than u, such that $v \notin Z(G)$ and $C(u) \cap C(v) = Z(G)$.
2. $C(v) = C(u)$ for all $v \in C(u) \setminus Z(G)$.
3. If $C(u) = C(v)$ then we do not count the contribution of u and v separately. All the $v \in C(u) \setminus Z(G)$ togetherly contribute the eigenvalue $|C(u)|$ of multiplicity $|C(u) \setminus Z(G)|$.
4. For any $w \in G \setminus C(u)$, $C(u) \cap C(w) = Z(G)$. Suppose not, then $C(u) = C(w)$ but it contradicts to the assumption $w \notin C(u)$.

Condition: either $C(u) = C(v)$ or $C(u) \cap C(v) = Z(G)$

Suppose that $u \in G \setminus Z(G)$ satisfies the condition.

1. There exists at least one $v \in G$, other than u, such that $v \notin Z(G)$ and $C(u) \cap C(v) = Z(G)$.
2. $C(v) = C(u)$ for all $v \in C(u) \setminus Z(G)$.
3. If $C(u) = C(v)$ then we do not count the contribution of u and v separately. All the $v \in C(u) \setminus Z(G)$ togetherly contribute the eigenvalue $|C(u)|$ of multiplicity $|C(u) \setminus Z(G)|$.
4. For any $w \in G \setminus C(u)$, $C(u) \cap C(w) = Z(G)$. Suppose not, then $C(u) = C(w)$ but it contradicts to the assumption $w \notin C(u)$.

Condition: either $C(u) = C(v)$ or $C(u) \cap C(v) = Z(G)$

Suppose that $u \in G \setminus Z(G)$ satisfies the condition.

1. There exists at least one $v \in G$, other than u, such that $v \notin Z(G)$ and $C(u) \cap C(v) = Z(G)$.
2. $C(v) = C(u)$ for all $v \in C(u) \setminus Z(G)$.
3. If $C(u) = C(v)$ then we do not count the contribution of u and v separately. All the $v \in C(u) \setminus Z(G)$ togetherly contribute the eigenvalue $|C(u)|$ of multiplicity $|C(u) \setminus Z(G)|$.
4. For any $w \in G \setminus C(u)$, $C(u) \cap C(w) = Z(G)$. Suppose not, then $C(u) = C(w)$ but it contradicts to the assumption $w \notin C(u)$.

Condition: either $C(u) = C(v)$ or $C(u) \cap C(v) = Z(G)$

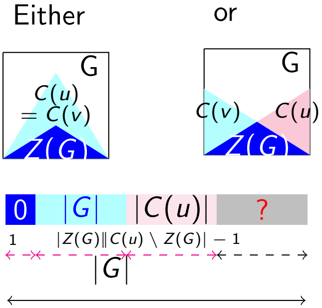
Suppose that $u \in G \setminus Z(G)$ satisfies the condition.

1. There exists at least one $v \in G$, other than u, such that $v \notin Z(G)$ and $C(u) \cap C(v) = Z(G)$.
2. $C(v) = C(u)$ for all $v \in C(u) \setminus Z(G)$.
3. If $C(u) = C(v)$ then we do not count the contribution of u and v separately. All the $v \in C(u) \setminus Z(G)$ togetherly contribute the eigenvalue $|C(u)|$ of multiplicity $|C(u) \setminus Z(G)|$.
4. For any $w \in G \setminus C(u)$, $C(u) \cap C(w) = Z(G)$. Suppose not, then $C(u) = C(w)$ but it contradicts to the assumption $w \notin C(u)$.

Condition: either $C(u) = C(v)$ or $C(u) \cap C(v) = Z(G)$

Suppose that $u \in G \setminus Z(G)$ satisfies the condition.

1. There exists at least one $v \in G$, other than u, such that $v \notin Z(G)$ and $C(u) \cap C(v) = Z(G)$.
2. $C(v) = C(u)$ for all $v \in C(u) \setminus Z(G)$.
3. If $C(u) = C(v)$ then we do not count the contribution of u and v separately. All the $v \in C(u) \setminus Z(G)$ togetherly contribute the eigenvalue $|C(u)|$ of multiplicity $|C(u) \setminus Z(G)|$.
4. For any $w \in G \setminus C(u)$, $C(u) \cap C(w) = Z(G)$. Suppose not, then $C(u) = C(w)$ but it contradicts to the assumption $w \notin C(u)$.



Non-Abelian Group

Let G be a non-abelian finite group with the following property: for all $u, v \in G \setminus Z(G)$

$$\text{either } C(u) = C(v) \text{ or } C(u) \cap C(v) = Z(G). \qquad (2)$$

Observation:

1. The condition given by Equation (2) is stronger than that of Equation (1).
2. If a group G satisfies Equation (2), each $C(u)$ is an abelian subgroup of G, for $u \in G \setminus Z(G)$.
3. If $Z(G)$ is trivial, then G with the above condition, turns out to be a centralizer abelian group [Suzuki, 1957, p. 686].

Non-Abelian Group

Let G be a non-abelian finite group with the following property: for all $u, v \in G \setminus Z(G)$

$$\text{either } C(u) = C(v) \text{ or } C(u) \cap C(v) = Z(G). \tag{2}$$

Observation:

1. The condition given by Equation (2) is stronger than that of Equation (1).
2. If a group G satisfies Equation (2), each $C(u)$ is an abelian subgroup of G, for $u \in G \setminus Z(G)$.
3. If $Z(G)$ is trivial, then G with the above condition, turns out to be a centralizer abelian group [Suzuki, 1957, p. 686].

Induced subgraph $\Gamma(G \setminus Z(G), E_G)$ of $\mathcal{C}_G = \Gamma(G, E)$

- $E = (G \setminus Z(G) \times G \setminus Z(G)) \cap E_G$.
- The graph $\Gamma(G \setminus Z(G), E_G)$ is obtained by removing all the vertices $v \in Z(G)$ and corresponding edges from $\mathcal{C}_G$.
- We refer the induced subgraph by $\Gamma(G \setminus Z(G), E)$.

Lemma

For any group G, satisfying the condition given by Equation (2), the graph $\Gamma(G \setminus Z(G), E)$ is disconnected.

- Suppose that the connected components of the graph $\Gamma(G \setminus Z(G), E)$ are denoted by $\{\mathcal{F}_i\}_{i=0}^{r-1}$, where $\mathcal{F}_i = (V(\mathcal{F}_i), E(\mathcal{F}_i))$.
- $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \emptyset$, for $i \neq j$ and $G = \cup_{i=0}^{r-1} V(\mathcal{F}_i) \cup Z(G)$.
- $V(\mathcal{F}_i) \cup Z(G) = C(u)$.

Induced subgraph $\Gamma(G \setminus Z(G), E_G)$ of $\mathcal{C}_G = \Gamma(G, E)$

- $E = (G \setminus Z(G) \times G \setminus Z(G)) \cap E_G$.
- The graph $\Gamma(G \setminus Z(G), E_G)$ is obtained by removing all the vertices $v \in Z(G)$ and corresponding edges from $\mathcal{C}_G$.
- We refer the induced subgraph by $\Gamma(G \setminus Z(G), E)$.

Lemma

For any group G, satisfying the condition given by Equation (2), the graph $\Gamma(G \setminus Z(G), E)$ is disconnected.

- Suppose that the connected components of the graph $\Gamma(G \setminus Z(G), E)$ are denoted by $\{\mathcal{F}_i\}_{i=0}^{r-1}$, where $\mathcal{F}_i = (V(\mathcal{F}_i), E(\mathcal{F}_i))$.
- $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \emptyset$, for $i \neq j$ and $G = \cup_{i=0}^{r-1} V(\mathcal{F}_i) \cup Z(G)$.
- $V(\mathcal{F}_i) \cup Z(G) = C(u)$.

Induced subgraph $\Gamma(G \setminus Z(G), E_G)$ of $\mathcal{C}_G = \Gamma(G, E)$

- $E = (G \setminus Z(G) \times G \setminus Z(G)) \cap E_G$.
- The graph $\Gamma(G \setminus Z(G), E_G)$ is obtained by removing all the vertices $v \in Z(G)$ and corresponding edges from $\mathcal{C}_G$.
- We refer the induced subgraph by $\Gamma(G \setminus Z(G), E)$.

Lemma

For any group G, satisfying the condition given by Equation (2), the graph $\Gamma(G \setminus Z(G), E)$ is disconnected.

- Suppose that the connected components of the graph $\Gamma(G \setminus Z(G), E)$ are denoted by $\{\mathcal{F}_i\}_{i=0}^{r-1}$, where $\mathcal{F}_i = (V(\mathcal{F}_i), E(\mathcal{F}_i))$.
- $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \emptyset$, for $i \neq j$ and $G = \cup_{i=0}^{r-1} V(\mathcal{F}_i) \cup Z(G)$.
- $V(\mathcal{F}_i) \cup Z(G) = C(u)$.

Induced subgraph $\Gamma(G \setminus Z(G), E_G)$ of $\mathcal{C}_G = \Gamma(G, E)$

- $E = (G \setminus Z(G) \times G \setminus Z(G)) \cap E_G$.
- The graph $\Gamma(G \setminus Z(G), E_G)$ is obtained by removing all the vertices $v \in Z(G)$ and corresponding edges from $\mathcal{C}_G$.
- We refer the induced subgraph by $\Gamma(G \setminus Z(G), E)$.

Lemma

For any group G, satisfying the condition given by Equation (2), the graph $\Gamma(G \setminus Z(G), E)$ is disconnected.

- Suppose that the connected components of the graph $\Gamma(G \setminus Z(G), E)$ are denoted by $\{\mathcal{F}_i\}_{i=0}^{r-1}$, where $\mathcal{F}_i = (V(\mathcal{F}_i), E(\mathcal{F}_i))$.
- $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \emptyset$, for $i \neq j$ and $G = \cup_{i=0}^{r-1} V(\mathcal{F}_i) \cup Z(G)$.
- $V(\mathcal{F}_i) \cup Z(G) = C(u)$.

Induced subgraph $\Gamma(G \setminus Z(G), E_G)$ of $\mathcal{C}_G = \Gamma(G, E)$

- $E = (G \setminus Z(G) \times G \setminus Z(G)) \cap E_G$.
- The graph $\Gamma(G \setminus Z(G), E_G)$ is obtained by removing all the vertices $v \in Z(G)$ and corresponding edges from $\mathcal{C}_G$.
- We refer the induced subgraph by $\Gamma(G \setminus Z(G), E)$.

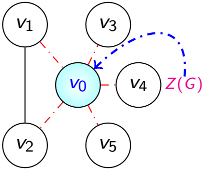
Lemma

For any group G, satisfying the condition given by Equation (2), the graph $\Gamma(G \setminus Z(G), E)$ is disconnected.

- Suppose that the connected components of the graph $\Gamma(G \setminus Z(G), E)$ are denoted by $\{\mathcal{F}_i\}_{i=0}^{r-1}$, where $\mathcal{F}_i = (V(\mathcal{F}_i), E(\mathcal{F}_i))$.
- $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \emptyset$, for $i \neq j$ and $G = \cup_{i=0}^{r-1} V(\mathcal{F}_i) \cup Z(G)$.
- $V(\mathcal{F}_i) \cup Z(G) = C(u)$.

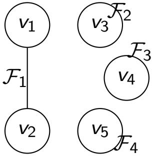
Induced subgraph $\Gamma(G \setminus Z(G), E_G)$ of $\mathcal{C}_G = \Gamma(G, E)$

- $E = (G \setminus Z(G) \times G \setminus Z(G)) \cap E_G$.
- The graph $\Gamma(G \setminus Z(G), E_G)$ is obtained by removing all the vertices $v \in Z(G)$ and corresponding edges from $\mathcal{C}_G$.
- We refer the induced subgraph by $\Gamma(G \setminus Z(G), E)$.

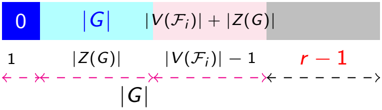


Lemma

For any group G, satisfying the condition given by Equation (2), the graph $\Gamma(G \setminus Z(G), E)$ is disconnected.



- Suppose that the connected components of the graph $\Gamma(G \setminus Z(G), E)$ are denoted by $\{\mathcal{F}_i\}_{i=0}^{r-1}$, where $\mathcal{F}_i = (V(\mathcal{F}_i), E(\mathcal{F}_i))$.
- $V(\mathcal{F}_i) \cap V(\mathcal{F}_j) = \emptyset$, for $i \neq j$ and $G = \cup_{i=0}^{r-1} V(\mathcal{F}_i) \cup Z(G)$.
- $V(\mathcal{F}_i) \cup Z(G) = C(u)$.



Laplacian spectra of $\Gamma(G \setminus Z(G), E)$

Remark

The Laplacian spectrum of the complete graph K_n is given by $\{0, n\}$, where the multiplicity of 0 and n are 1 and $n-1$, respectively. Consequently, for the disjoint union of the complete graphs $\Gamma = \oplus_{i=1}^{k} K_{m_i}$, the Laplacian spectrum is given by $0, m_1, \ldots, m_k$ with multiplicity $k, m_1 - 1, \ldots, m_k - 1$, respectively.

Note:

1. Note that each $\mathcal{F}_i$ is a complete graph K_{m_i}, where $m_i = |C(u_i)| - |Z(G)|$.
2. Therefore, $\Gamma(G \setminus Z(G), E) = \oplus_{i=0}^{r-1} K_{m_i}$ and Laplacian spectra of $\Gamma(G \setminus Z(G), E)$ can be calculated by the above remark.
3. If the center $Z(G)$ is trivial, then the Laplacian spectrum of $\Gamma(G \setminus Z(G), E)$ is discussed in [Dutta and Nath, 2018].

Henceforth, we use the notation $\mathcal{F}_i$ to refer both the connected component and the vertex set $V(\mathcal{F}_i)$.

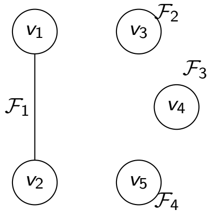


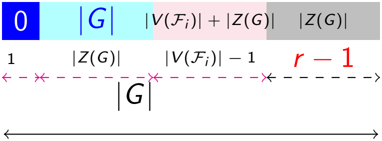
Figure: $\Gamma(\mathfrak{S}_3 \setminus Z(\mathfrak{S}_3), E)$

$\sigma(L_\Gamma) = \{0(4), 2(1)\}$

Non-Abelian Group

Theorem

Let G be a non-abelian group with the property described in Equation (2) and the Laplacian matrix associated to $\mathcal{C}_G$ be denoted by L. The matrix L has an eigenvalue $|Z(G)|$ with the multiplicity at least $r-1$, where r is the number of connected components of the graph $\Gamma(G \setminus Z(G), E)$.



Back to complete Laplacian spectra

The complete Laplacian spectra of $\mathcal{C}_G$ for non-abelian G

0	1
$Z(G)$	$(r-1)$, where $r =$ number of connected components of the graph $\Gamma(G \setminus Z(G), E)$.
λ_i, where $\lvert C(v)\rvert = \lvert C(u)\rvert = \lambda_i$ for every $u, v \in \mathcal{F}_i$,	$\lvert\mathcal{F}_i\rvert - 1$
$\lvert G\rvert$	$\lvert Z(G)\rvert$

The complete Laplacian spectra of $\mathcal{C}_G$ for non-abelian G

0	1
$Z(G)$	$(r-1)$, where $r =$ number of connected components of the graph $\Gamma(G \setminus Z(G), E)$.
λ_i, where $\lvert C(v)\rvert = \lvert C(u)\rvert = \lambda_i$ for every $u, v \in \mathcal{F}_i$,	$\lvert\mathcal{F}_i\rvert - 1$
$\lvert G\rvert$	$\lvert Z(G)\rvert$

The complete Laplacian spectra of $\mathcal{C}_G$ for non-abelian G

0	1
$Z(G)$	$(r-1)$, where $r =$ number of connected components of the graph $\Gamma(G \setminus Z(G), E)$.
λ_i, where $\lvert C(v)\rvert = \lvert C(u)\rvert = \lambda_i$ for every $u, v \in \mathcal{F}_i$,	$\lvert\mathcal{F}_i\rvert - 1$
$\lvert G\rvert$	$\lvert Z(G)\rvert$

The complete Laplacian spectra of $\mathcal{C}_G$ for non-abelian G

0	1
$Z(G)$	$(r-1)$, where $r =$ number of connected components of the graph $\Gamma(G \setminus Z(G), E)$.
λ_i, where $\lvert C(v)\rvert = \lvert C(u)\rvert = \lambda_i$ for every $u, v \in \mathcal{F}_i$,	$\lvert\mathcal{F}_i\rvert - 1$
$\lvert G\rvert$	$\lvert Z(G)\rvert$

The complete Laplacian spectra of $\mathcal{C}_G$ for non-abelian G

0	1
$Z(G)$	$(r-1)$, where $r =$ number of connected components of the graph $\Gamma(G \setminus Z(G), E)$.
λ_i, where $\lvert C(v)\rvert = \lvert C(u)\rvert = \lambda_i$ for every $u, v \in \mathcal{F}_i$,	$\lvert\mathcal{F}_i\rvert - 1$
$\lvert G\rvert$	$\lvert Z(G)\rvert$

Non-Abelian Group

• 0	• 1
• $Z(G)$	• $(r-1)$, where $r =$ number of connected components of the graph $\Gamma(G \setminus Z(G), E)$.
• λ_i, where $\lvert C(v)\rvert = \lvert C(u)\rvert = \lambda_i$ for every $u, v \in \mathcal{F}_i$,	• $\lvert\mathcal{F}_i\rvert - 1$
• $\lvert G\rvert$	• $\lvert Z(G)\rvert$

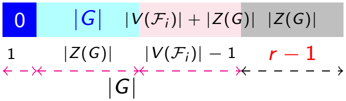
$$
\begin{aligned}
|G| &= |Z(G)| + \sum_{i=1}^{r} |\mathcal{F}_i| \\
&= |Z(G)| + \sum_{i=1}^{r} (|\mathcal{F}_i| - 1) + r \\
&= |Z(G)| + \sum_{i=1}^{r} (|\mathcal{F}_i| - 1) + (r-1) + 1
\end{aligned}
$$

Necessary conditions for being a commuting graph

Theorem

Let Γ be a graph with the Laplacian spectrum $0 = \lambda_1 < \lambda_2 < \ldots < \lambda_r = |G|$ with multiplicity $m_1, m_2, \ldots, m_r$, respectively. There does not exist any finite group G (that satisfies Equation (2)) such that $\Gamma = \mathcal{C}_G$ if the Laplacian spectrum fails to satisfy any of the following conditions.

1. *The multiplicity of λ_1 is $m_1 = 1$.*
2. *The number of distinct eigenvalues r is greater equal to 4.*
3. *The algebraic connectivity of Γ is the multiplicity of the largest eigenvalue, that is, $\lambda_2 = m_r$.*
4. *The algebraic connectivity λ_2 divides λ_j for all $j = 2, 3 \ldots, r$.*
5. *The cardinality of the set of vertices $\sum_{i=1}^{r} m_r = \lambda_r$.*
6. *Any nonzero eigenvalue of the Laplacian matrix of Γ divides the largest eigenvalue λ_r.*
7. *Each λ_i is a non-negative integer for all $i = 1, \ldots, r$.*



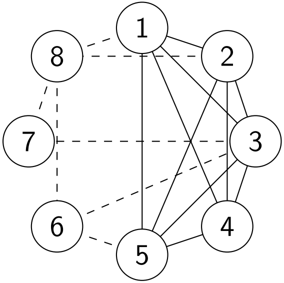
Graph Invariants for Commuting Graphs

Definition

A clique S in a graph $\Gamma(V,E)$ is a subset $S \subseteq V$ such that any two distinct vertices $v_1, v_2 \in S$ are adjacent to each other. The cardinality of the maximum clique is called the clique number, $\omega(\Gamma)$.

Clearly, $\omega(\mathcal{C}_G) = |G|$, whenever G is an abelian group. For any non-abelian group G we have the following.

- if $u \in G \setminus Z(G)$ then $C(u)$ is a clique in $\mathcal{C}_G$.
- $\omega(\mathcal{C}_G) = \max_{u \in G \setminus Z(G)} |C(u)|$.



Clique Number

Since $|C(u)|$ is an eigenvalue of $L_{\mathcal{C}_G}$ for all $u \in G \setminus Z(G)$,

Proposition

If G is a non-abelian group satisfying Equation 2, then the clique number $\omega(\mathcal{C}_G)$ is given by

(i) *the second largest eigenvalue of the Laplacian matrix of the commuting graph $\mathcal{C}_G$, if $C(G) \supsetneq Z(G)$ and*

(ii) $|Z(G)| + 1$, *if* $Z(G) = C(G)$,

where $C(G) = \{v \in G : d(v) - |Z(G)| > 0\}$.

Mean distance and Graph diameter

- The distance between two vertices of a graph is the number of edges in a shortest path connecting them.
- The graph distance matrix is the square matrix $(\gamma_{ij})_{i,j=1}^{|G|}$.
- $\gamma_{ij} =$ The distance between the vertices v_i and v_j.
- The diameter of the graph is the maximum element of the graph distance matrix.
- The mean distance is the average of all elements of the graph distance matrix.
- We have proved that

$$\text{The mean distance of } \mathcal{C}_G = \frac{2|G|^2 - 2|G| - \sum_{v \in G} d(v)}{|G|^2}.$$

Separation Problems

Definition

[Cvetković et al., 2010, p. 199] Let $\Gamma(V, E)$ be a graph. For any $S \subset V$, the collection of all the edges that contains vertices from both the sets S and S^c $(= V \setminus S)$ are called the *edge boundary of the set* S. The edge boundary of S is denoted by ∂S.

In the next result we give a range of $\frac{|\partial S|}{|S||S^c|}$.

Proposition

Let G be a non-abelian group with the property described in Equation (2) and $\mathcal{C}_G$ be the commuting graph associated to the group G. For any $S \subset G$, the edge boundary ∂S of S in the graph $\mathcal{C}_G$ satisfies the following inequality:

$$\frac{|Z(G)|}{|G|} \leq \frac{|\partial S|}{|S||S^c|} \leq 1,$$

where $S^c = G \setminus S$.

Bipartition Width

The *bipartition width* of a graph $\Gamma(V, E)$ is $bw(G) := \min\{|\partial S| : S \subset V, |S| = \lfloor \frac{|V|}{2} \rfloor\}$, where $\lfloor \frac{|V|}{2} \rfloor$ is the greatest integer less than $\frac{|V|}{2}$, see [Cvetković et al., 2010, p. 200]. Then, the next result is an immediate consequence of Proposition 2.

Corollary

If G be a non-abelian group with the property described in Equation (2) then

$$bw(\mathcal{C}_G) \geq \begin{cases} \frac{|G||Z(G)|}{4}, & \text{if } |G| \text{ is even,} \\ \frac{(|G|^2-1)|Z(G)|}{4|G|}, & \text{if } |G| \text{ is odd.} \end{cases}$$

A Lemma

Lemma

Let G be a non-abelian group and $\mathcal{C}_G$ be the commuting graph of G and S be a subset of G.

A. *If S satisfies any one of the following conditions, then* $\frac{|\partial S|}{|S|} \geq |Z(G)|$.
   1. $S \subset G$ *be such that* $S \cap Z(G) = \emptyset$,
   2. $Z(G) \subseteq S$ *with* $0 < |S| \leq \frac{|G|}{2}$,
   3. $Z(G) \supseteq S$ *with* $0 < |S| \leq \frac{|G|}{2}$.

B. *Moreover, suppose that* $S \cap Z(G) \neq \emptyset$ *and* $Z(G) \setminus S \neq \emptyset$, *then*

$$|\partial S| = |S||Z(G)| + |Z(G) \cap S|(|G| - 2|S| - |Z(G) \setminus S|) + \sum_{u \in S \setminus Z(G)} |\mathcal{F}_u \setminus S|,$$

where $\mathcal{F}_u = C(u) \setminus Z(G)$ *for an element* $u \in G \setminus Z(G)$.

Isoperimetric Number

- We recall the definition of *isoperimetric number* $i(\Gamma)$ of a graph $\Gamma(V, E)$ following [Cvetković et al., 2010, p. 205]. The isoperimetric number $i(\Gamma)$ is given by

$$i(\Gamma) := \min_{0<|S|<\frac{|V|}{2}} \frac{|\partial S|}{|S|}. \tag{3}$$

- If Γ is a graph, except the complete graph K_n with order $n = 1, 2, 3$, then in [Mohar, 1989, p. 283, Theorem 4.1, Theorem 4.2] a range for $i(\Gamma)$ is given and that is

$$\frac{\lambda_2}{2} \leq i(\Gamma) \leq \sqrt{\lambda_2(2\Delta - \lambda_2)},$$

where λ_2 is the second least element of the Laplacian spectrum of Γ and Δ is the maximum degree of vertex in Γ.

Isoperimetric Number of C_G

Proposition

If G be a non-abelian group with the property described in Equation (2) *then*

$$\frac{|Z(G)|}{2} \leq i(\mathcal{C}_G) \leq \sqrt{|Z(G)|(2(|G|-1)-|Z(G)|)}.$$

Theorem

Let G be a finite group. Then the following hold:

A. *If G is abelian, then* $i(\mathcal{C}_G) = \lceil \frac{|G|}{2} \rceil$, *where* $\lceil \frac{|G|}{2} \rceil$ *denotes the least integer greater than* $\frac{|G|}{2}$.

B. *If G is a non-abelian group that satisfies Equation* (2).

1. *If G has a trivial center, then* $i(\mathcal{C}_G) = 1$.
2. *Suppose* $|Z(G)| = 2$. *If* $|C(u) \setminus Z(G)| = l$ *for all* $u \in G \setminus Z(G)$, *where* $l < \frac{|G|}{2}$ *and* l *does not divide* $\frac{|G|}{2} - 1$, *then* $i(\mathcal{C}_G) = 2$.

Thank You

Proof..

Lemma

$G \implies$ a non-abelian group

- $0 \in \sigma(L_{\mathcal{C}_G})$ *Multiplicity* $= 1$ *Eigenvector* $= \chi_V = (1, \ldots, 1)^t$
- $|G| \in \sigma(L_{\mathcal{C}_G})$ *Multiplicity at least* $|Z(G)|$.

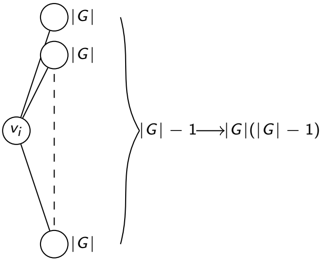
- The graph $\mathcal{C}_G$ is connected. Therefore, the first part follows.
- For the 2^{nd} part, we enumerate $Z(G) = \{v_1, \ldots, v_m\}$. For each $v_i \in Z(G)$, we consider $y_i \in \mathbb{R}^V$ defined as

$$y_i(v) = \begin{cases} |G| - 1 & \text{if } v = v_i, \\ -1 & \text{otherwise.} \end{cases}$$

- $L_{\mathcal{C}_G} y_i = |G| y_i$ and $\{y_i\}_{i=1}^m$ is linearly independent.

back

$$(L_\Gamma y_i)(u) = \sum_{v \in V} a_{uv}(y_i(u) - y_i(v)).$$



Reason...

There exists at least one $v \in G$, other than u, such that $v \notin Z(G)$ and $C(u) \cap C(v) = Z(G)$.

If not, arguing by contradiction, it follows from Equation (1) that $C(u) = C(v)$ for every $v \in G \setminus Z(G)$. Equivalently, $u \in C(v)$ for every $v \in G$. That is, $C(u) = G$ which is a contradiction to the assumption $u \in G \setminus Z(G)$. back

Reason...

$C(v) = C(u)$ for all $v \in C(u) \setminus Z(G)$.

- Either $C(u) = C(v)$ or $C(u) \cap C(v) = Z(G)$.
- $v \in C(u) \setminus Z(G) \implies v \in C(u) \cap C(v)$
- Since, $v \notin Z(G)$ and $v \in C(u) \cap C(v)$, therefore, $C(u) \cap C(v) \neq Z(G)$.
- Therefore, $C(v) = C(u)$

back

Reason...

$C(u)$ is an abelian subgroup of G, whenever $u \in G \setminus Z(G)$ satisfies Equation (1).

Suppose that $x, y \in C(u)$ then the following possibilities arise:

1. If one of x or y belongs to $Z(G)$, then $xy = yx$.
2. If $x, y \notin Z(G)$ then $x \in (C(u) \cap C(x)) \setminus Z(G)$ and $y \in (C(u) \cap C(y)) \setminus Z(G)$. By Equation (2), $C(x) = C(u) = C(y)$ and hence $xy = yx$.

Therefore, $C(u)$ is an abelian subgroup of G, for $u \in G \setminus Z(G)$ satisfies Equation (2). back

References I

Cvetković, D., Rowlinson, P., and Simić, S. (2010).
An introduction to the theory of graph spectra, volume 75 of *London Mathematical Society Student Texts*.
Cambridge University Press, Cambridge.

Dutta, J. and Nath, R. K. (2018).
Laplacian and signless Laplacian spectrum of commuting graphs of finite groups.
Khayyam J. Math., 4(1):77–87.

Mohar, B. (1989).
Isoperimetric numbers of graphs.
J. Combin. Theory Ser. B, 47(3):274–291.

References II

Suzuki, M. (1957).
The nonexistence of a certain type of simple groups of odd order.
Proc. Amer. Math. Soc., 8:686–695.