The Atiyah-Jänich Theorem

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Overview

- Fredholm Theory
- K-Theory
- Construction of the Index Bundle
- The Atiyah-Jänich Theorem

Goal!

Theorem (Atiyah-Jänich)

Let X be a compact Hausdorff space. Then $\exists$ a semi-group isomorphism

$$[X, \mathcal{F}(\mathcal{H})] \cong \mathrm{K}(X).$$

Fredholm Index

Let $\mathcal{H}$ be a separable Hilbert space.

Fredholm Operators

Let $\mathcal{F}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) : \dim \ker(T) < \infty \text{ and } \dim \operatorname{coker}(T) < \infty\}$

Definition (Fredholm Index)

The Fredholm Index is a map **ind** : $\mathcal{F}(\mathcal{H}) \to \mathbb{Z}$ such that

$$\operatorname{ind}(T) = \operatorname{dimker}(T) - \operatorname{dimcoker}(T).$$

- Let $T_1, T_2 \in \mathcal{F}(\mathcal{H})$, then $\operatorname{ind}(T_1 T_2) = \operatorname{ind}(T_1) + \operatorname{ind}(T_2)$.
- $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$.
- $\operatorname{ind} : \mathcal{F}(\mathcal{H}) \to \mathbb{Z}$ is surjective and locally constant.

Vector Bundles on Compact Hausdorff Spaces

Definition

A $\mathbb{C}$-vector bundle is a triple (E, X, p) where $p : E \to X$ is a surjective map with X being a compact Hausdorff space $\ni$

- The fiber of $x \in X$, $E_x := p^{-1}(x)$ is a $\mathbb{C}-$vector space,
- $\forall x \in X \ \exists U \subset X$ (open) $\ni x \in U$ and $\phi : p^{-1}(U) \xrightarrow{\cong} U \times \mathbb{C}^n$ where $\phi|_{E_x}$ is a fibre-preserving $\mathbb{C}-$linear isomorphism for some $n \in \mathbb{N}$. (This is called the local triviality condition).

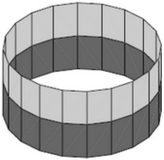

Figure: The trivial bundle $\varepsilon^1 \cong \mathbb{S}^1 \times \mathbb{R}$

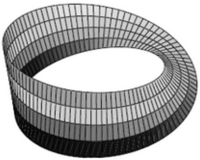

Figure: The Möbius bundle over $\mathbb{S}^1$

Definition of K(X)

Definition (Isomorphism of Vector Bundles)

Let $p_1 : E_1 \to X$ and $p_2 : E_1 \to X$ be two vector bundles. They are isomorphic if there exists a map $\phi : E_1 \to E_2$ s.t. this diagram commutes,

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ & \searrow^{p_1} & \downarrow{\scriptstyle p_2} \\ & & X \end{array}$$

and $\phi_x : E_1|_{p_1^{-1}(x)} \to E_2|_{p_2^{-1}(x)}$ is a linear isomorphism.

Vect(X) := Isomorphism classes of Vector Bundles over X.

Notation: $\varepsilon^k := X \times \mathbb{C}^k$ is called the trivial bundle of rank $k \in \mathbb{N} \cup \{0\}$.

K$(X) := \{[\varepsilon^n]_s - [E]_s : \varepsilon^n$ is the trivial bundle of rank $n\}$

where $E_1 \cong_s E_2$ if $E_1 \oplus \varepsilon^k \cong E_2 \oplus \varepsilon^k$ for some $k \in \mathbb{N} \cup \{0\}$.

Construction of the Index Bundle - 1

Lemma

Let $T \in \mathcal{F}(\mathcal{H})$ and $\mathcal{H}_1 \subset \mathcal{H}$ be a closed subspace of finite codimension such that $\mathcal{H}_1 \cap \mathrm{N}(T) = \{0\}$. Then $T(\mathcal{H}_1)$ is closed and $\mathcal{H}/T(\mathcal{H}_1)$ is finite dimensional.

Proof.

$0 \to T(\mathcal{H})/T(\mathcal{H}_1) \to \mathcal{H}/T(\mathcal{H}_1) \to \mathrm{coker}(T) \to 0$ □

Lemma

There is an open neighbourhood Ω of $T \in \mathcal{F}(\mathcal{H})$ such that for all $S \in \Omega$

- $\mathcal{H}_1 \cap N(S) = \{0\}$, *$S(\mathcal{H}_1)$ is closed,*
- $\cup_{S \in \Omega} \mathcal{H}/S(\mathcal{H}_1)$ *is a trivial vector bundle.*

Construction of the Index Bundle - 2

Theorem

Let X be a compact Hausdorff space and let $\phi : X \to \mathcal{F}$ be a continuous map. Then

- $\exists \mathcal{H}_1 \subset \mathcal{H}$ *be a closed subspace of finite codimension such that for all* $x \in X$
$$\mathcal{H}_1 \cap \mathrm{N}(\phi_x) = \{0\}.$$
- $\cup_{x \in X} \mathcal{H}/\phi_x(\mathcal{H}_1)$ *is a vector bundle over* X.

Proof.

- For every $x \in X$ choose $\mathcal{H}_{1x} := \mathrm{N}(\phi_x)^{\perp}$ and fix open nbds Ω_x.
- We can cover X by a finite $\{\Omega_{x_1}, ..., \Omega_{x_k}\}$.
- Let $\mathcal{H}_1 := \cap_{i=1}^{k} \mathcal{H}_{1x_i}$. Then $\cup_{x \in X} \mathcal{H}/\phi_x(\mathcal{H}_1)$ is a vector bundle.

□

Definition of the Index Bundle

Definition

The index bundle associated to a map $\phi : X \to \mathcal{F}(\mathcal{H})$ is

$$\text{ind}(\phi) = [\mathcal{H}/\mathcal{H}_1] - [\mathcal{H}/\phi(\mathcal{H}_1)] \in \mathrm{K}(X)$$

where $[\mathcal{H}/\mathcal{H}_1]$ is the trivial bundle $X \times (\mathcal{H}/\mathcal{H}_1)$ and $[\mathcal{H}/\phi(\mathcal{H}_1)]$ is the bundle $\cup_{x \in X} \mathcal{H}/\phi_x(\mathcal{H}_1)$.

- If $\phi_1, \phi_2 : X \to \mathcal{F}(\mathcal{H})$ are homotopic, then $\text{ind}(\phi_1) = \text{ind}(\phi_2)$.
- Hence, ind: $[X, \mathcal{F}(\mathcal{H})] \to \mathrm{K}(X)$.
- ind is a homomorphism of semigroups.

Example

Let $X = \{*\}$ and let $\phi : \{*\} \mapsto T \in \mathcal{F}(\mathcal{H})$.
Then we have $\text{ind}(\phi) = [X \times \ker(T)] - [X \times \text{coker}(T)]$.

The Atiyah-Jänich Theorem

Theorem (Atiyah-Jänich)

Let X be a compact Hausdorff space. Then $\exists$ a semi-group isomorphism

$$\text{ind} : [X, \mathcal{F}(\mathcal{H})] \xrightarrow{\cong} \mathrm{K}(X).$$

Lemma (Kuiper's Theorem)

The group of invertible elements of $\mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$ is contractible to the identity map.

Proof.

$$[X, \mathcal{F}(\mathcal{H})^{\times}] \xrightarrow{i} [X, \mathcal{F}(\mathcal{H})] \xrightarrow{\text{ind}} \mathrm{K}(X) \to 0$$

□

References

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[2] G. J. Murphy, *C*-algebras and operator theory.* Academic press, 2014.

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